The anisotropic multichannel spin-S Kondo model: Calculation of scales from a novel exact solution

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Abstract. A novel exact solution of the multichannel spin-S Kondo model is presented, based on a lattice path integral approach of the single channel spin-1/2 case. The spin exchange between the localized moment and the host is of XXZ-type, including the isotropic XXX limit. The free energy is given by a finite set of non-linear integral equations, which allow for an accurate determination of high- and low-temperature scales.

PACS. 72.15.Qm Scattering mechanisms and Kondo effect – 04.20.Jb Exact solutions – 75.20.Hr Local moment in compounds and alloys; Kondo effect, valence fluctuations, heavy fermions – 75.10.Lp Band and itinerant models

1 Introduction

The isotropic multichannel spin-S Kondo model describes the XXX-like spin scattering between non-interacting spin-1/2 fermions in m channels and a single localized spin-S impurity. It has been proposed by Nozières and Blandin [1] to account for the orbital structure of spin-S impurities in metals. Physical realizations are discussed in the review article [2]. An exact solution to this model was found by Tsvelick and Wiegmann [3–5] using the Bethe Ansatz (BA). By applying the thermodynamic Bethe Ansatz (TBA), the thermodynamics has been calculated by Tsvelick [6] and Andrei and Destri [7]. In the TBA approach, the free energy is encoded by infinitely many coupled non-linear integral equations. Recently, Schlottmann [8,9] and Zaránd et al. [10] obtained the free energy for the anisotropic multichannel spin-S Kondo model with XXZ-like exchange in the TBA approach. Schlottmann observed a quantum critical point for the overscreened model in the limit of low temperatures, Zaránd et al. focused on the underscreened two-channel model. Here, we present a novel exact solution in which the free energy is encoded in a *finite* set of non-linear integral equations (NLIE), including both the anisotropic and isotropic models for arbitrary S and m. On the one hand, we confirm known results by this novel solution, on the other hand, we obtain farther reaching results as far as ratios of high- and low-temperature scales are concerned.

In a recent publication [11], we proposed a lattice path integral approach to an Anderson-like impurity in a cor-

related host. We obtained this model from the Hamiltonian limit of a gl(2|1) symmetric transfer matrix. The free energy contributions of both the host and the impurity were calculated exactly by combining the Trotter Suzuki mapping with the quantum transfer matrix technique. We showed that the model allows for the Kondo limit, i.e. a localized magnetic impurity in a free host with linearized energy-momentum relation. In the Kondo limit, the symmetry of the coupling between host and impurity is reduced from gl(2|1) to su(2). In the following, we exploit this observation to propose a $U_q$su(2) symmetric quantum transfer matrix (QTM), whose largest eigenvalue yields the free energy contribution of the impurity in the thermodynamic limit, in close analogy to [11]. The q-deformation accounts for an XXZ-like spin exchange between the localized moment and the host, including the isotropic limit $q \to 1$. The transfer matrix is built from local R-operators, each acting in the tensor product space of two $U_q$su(2) modules, carrying an $(m+1)$-dimensional irrep and an $(l+1)$-dimensional irrep of $U_q$su(2), respectively. Here m denotes the number of the electronic host channels and $l = 2S$ is twice the impurity spin. By making use of analyticity arguments, the free energy is represented by $(\max[l, m] + 1)$-many NLIE. These equations are investigated both analytically and numerically for the accurate calculation of scales in the limits of low and high temperatures.

This article is organized as follows. In the next section, the free energy contribution of the impurity is determined as the largest eigenvalue of a certain QTM. The third and fourth sections are devoted to the calculation of thermodynamic equilibrium functions and scales in the limits of

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low and high temperatures, respectively. In the last section, we summarize the results.

In all that follows we set $k_B = 1$, and $g\mu_B = 1$, where k_B is Boltzmann's constant, g is the gyromagnetic factor and μ_B is the Bohr magneton. An index i (h) denotes quantities pertaining to the impurity (host).

2 Calculation of the free energy

Let $V_q^{(l)}$ be the module carrying the l-dimensional irrep of $\mathrm{U}_q\mathrm{su}(2)$; in the limit $q = 1$, $V_1^{(l)}$ carries the l-dimensional irrep of su(2). Consider the matrices $R_q^{(l,m)}(x) \in \mathrm{End}\left(V_q^{(l)} \otimes V_q^{(m)}\right)$ constructed such that the Yang-Baxter-Equation (YBE) is fulfilled:

$$\left[R_q^{(l,m)}(x)\right]_{\beta',\gamma'}^{\beta,\gamma} \left[R_q^{(n,m)}(y)\right]_{\alpha',\gamma''}^{\alpha,\gamma'} \left[R_q^{(n,l)}(y-x)\right]_{\alpha'',\beta''}^{\alpha',\beta'} = \left[R_q^{(n,l)}(y-x)\right]_{\alpha',\beta'}^{\alpha,\beta} \left[R_q^{(n,m)}(y)\right]_{\alpha'',\gamma'}^{\alpha',\gamma} \left[R_q^{(l,m)}(x)\right]_{\beta'',\gamma''}^{\beta',\gamma'}.$$

The $R_q^{(l,m)}(x)$ are obtained by fusing a lattice of $l \times m$ many operators $R_q^{(1,1)}$. Then $V_q^{(k)}$ is the subspace of completely symmetric tensors in $\left[V_q^{(1)}\right]^{\otimes k}$, with $k = l, m$. In [12], the explicit expression of $R_q^{(l,m)}(u)$ is given. If in the isotropic single channel case, the spectral parameter is chosen to take the special value $x_0 := \left[\frac{2}{l+1} \tan J(l+1)\right]^{-1}$, the matrix $R_1^{(l,1)}(x_0)$ simplifies to [13]

$$R_1^{(l,1)}(x_0) = \exp\left[\mathrm{i}2J\mathbf{S}\cdot\boldsymbol{\sigma}\right]. \tag{1}$$

The logarithm of equation (1) is the spin exchange operator between the impurity spin $S = l/2$ and one electron, where the spin-exchange coupling J is parameterized by x_0.

It has been argued by Tsvelick et al. [3] that the exchange in the multichannel Kondo model is equivalent to an exchange model for spin-$m/2$ electrons scattered by the spin-S impurity. This equivalence has been proven by the equality of the exact solutions of both models at zero temperature. We expect that more insight into this question is possible by a lattice path integral formulation in analogy to the $S = 1/2$, $m = 1$, $\gamma = 0$ case [11]. We leave this task as a future challenge. Here, we rely on the above mentioned equivalence, so that the Hamiltonian density is given by:

$$\mathcal{H} = \mathcal{H}_h + \mathcal{H}_{sd} + \mathcal{H}_{ex}$$

$$\mathcal{H}_h = -\mathrm{i}v_f \sum_{\sigma=\pm 1/2} \sum_{k=1}^{m} :\psi_{\sigma,k}^\dagger(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\sigma,k}(x): \tag{2a}$$

$$\mathcal{H}_i = -\mathrm{i}\delta(x) \sum_{\sigma,\sigma'} :\psi_\sigma^\dagger(x)\left[\ln R_q^{(l,m)}(x_0)\right]_\sigma^{\sigma'}\psi_{\sigma'}(x): \tag{2b}$$

$$\mathcal{H}_{ex} = \frac{h}{2}\left[\sum_{\sigma=-m/2}^{m/2}\sigma n_\sigma(x) + \delta(x)\sum_{\tau=-S}^{S}\tau n_{d,\tau}\right] \tag{2c}$$

$$n_{\sigma,k} := :\psi_{\sigma,k}^\dagger\psi_{\sigma,k}:, \quad n_k = \sum_\sigma n_{\sigma,k}, \quad n_\sigma = \sum_k n_{\sigma,k},$$

where x_0 and q parameterize the coupling constants. In the following, q is given by a real constant γ, $q =: \exp(\mathrm{i}\gamma)$, such that $\gamma \neq 0$ induces an Ising-like anisotropy, $J_\perp < J_\parallel$. The matrix $R_q^{(l,m)}$ acts in the electronic space with operator-valued entries in the impurity space. Only the totally symmetric combination of spinors, with total spin $m/2$, interacts with the impurity, so that the spin indices in equation (2b) are $\sigma, \sigma' = -m/2, \ldots, m/2$.

Consider the monodromy matrix

$$T_q^{(l,m)} := \left[R_q^{(l,m)}\right]_{a,N}\left[R_q^{(l,m)}\right]_{a,N-1}\cdots\left[R_q^{(l,m)}\right]_{a,1}$$

$$\tau_q^{(l,m)} = \mathrm{tr}_a T_q^{(l,m)}, \tag{3}$$

where $\left[R_q^{(l,m)}\right]_{a,n}$ acts in the tensor product $V_a^{(l)} \otimes V_n^{(m)}$. In the last line the transfer matrix $\tau_q^{(l,m)}$ is defined as the trace over the auxiliary space of $T_q^{(l,m)}$. In view of the definition of the Hamiltonian in equation (2b), the auxiliary space a is identified with the impurity space and the quantum spaces $1,\ldots,N$ correspond to the electrons. We will now show that under the substitution $N \to \mathrm{i}\beta D$, the largest eigenvalue $\Lambda_1^{(1,1)}(x_0)$ of $T_1^{(1,1)}(x_0)$ is directly related to the free energy contribution of the impurity f_i found in [11],

$$-\beta f_i = \ln \Lambda_1^{(1,1)}(x_0)\Big|_{N\to\mathrm{i}\beta D}. \tag{4}$$

Here D has the meaning of a bandwidth parameter of the host. A magnetic field is introduced by twisted boundary conditions.

The largest eigenvalue in equation (4) is expressed through auxiliary functions $\mathfrak{b}, \overline{\mathfrak{b}}$ ($\mathfrak{B} = 1 + \mathfrak{b}$, $\overline{\mathfrak{B}} = 1 + \overline{\mathfrak{b}}$) in analogy to [14]. After the substitution $N \to \mathrm{i}\beta D$, these functions are given by the non-linear integral equation

$$\ln\mathfrak{b}(x) = -2D\beta\arctan\mathrm{e}^x + \frac{\beta h}{2} + \left[\kappa * \ln\mathfrak{B} - \kappa_- * \ln\overline{\mathfrak{B}}\right](x), \tag{5}$$

where $\kappa(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\pi/2|k|}}{2\cosh\frac{\pi k}{2}}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$ and $\kappa_\pm(x) := \kappa(x \pm \mathrm{i}\pi)$. An equation similar to (5) holds with $\mathfrak{b} \to \overline{\mathfrak{b}}$, $h \to -h$ and $\kappa_- \to \kappa_+$. The free energy is obtained from the auxiliary functions by

$$-\beta f_i = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(x+x_0)}\ln[\mathfrak{B}\overline{\mathfrak{B}}](x)\mathrm{d}x.$$

These equations are equivalent to the corresponding TBA equations [13], as can be shown by fusion techniques described in [15]. Proceeding as in [13], the universal limit $D \to \infty$ is taken by defining the temperature scale

$$T_K := 2D\mathrm{e}^{-\pi x_0}. \tag{6}$$

This results in the D-independent equations

$$\ln \mathfrak{b}(x) = -\mathrm{e}^x + \frac{\beta h}{2} + \left[\kappa * \ln \mathfrak{B} - \kappa_- * \ln \overline{\mathfrak{B}} \right](x)$$

$$-\beta f_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\cosh(x + T/T_K)} \ln[\mathfrak{B}\overline{\mathfrak{B}}](x)\mathrm{d}x,$$

in complete agreement with results from [11]. Thus we argue that the impurity contribution to the free energy of the anisotropic multichannel spin-S Kondo model is given by

$$-\beta f_i = \ln \Lambda_q^{(l,m)}(x_0)\Big|_{N \to \mathrm{i}\beta D}, \tag{7}$$

where the universal limit $D \to \infty$ is implicit.

The eigenvalue $\Lambda_q^{(l,m)}$ is found by standard Bethe-Ansatz techniques [12,16]. Following [17], one introduces suitably chosen auxiliary functions in order to exploit the analyticity properties of the largest eigenvalue. We restrict ourselves to the anisotropy parameter

$$\gamma < \frac{\pi}{2n} =: \gamma_{\max}, \tag{8}$$

where $n := \max(l, m)$. We define a temperature scale

$$T_K = 2D\mathrm{e}^{-\pi x_0/\gamma}, \tag{9}$$

from which the isotropic case (6) is recovered by scaling $x_0 \to \gamma x_0$. One is left with the following system of NLIE for the auxiliary functions $y_j = 1 + Y_j$, $(j = 1, \ldots, n-1)$, $\mathfrak{b}_n = 1 + \mathfrak{B}_n$ and $\overline{\mathfrak{b}}_n = 1 + \overline{\mathfrak{B}}_n$:

$$\mathbf{y}(x) = \mathbf{d}(x) + [\hat{\kappa} * \mathbf{Y}](x) \tag{10}$$

$$\mathbf{y} := \left(\ln y_1, \ln y_2, \cdots, \ln y_{n-1}, \ln \mathfrak{b}_n, \ln \overline{\mathfrak{b}}_n \right)^T$$

$$\mathbf{Y} := \left(\ln Y_1, \ln Y_2, \cdots, \ln \mathfrak{B}_n, \ln \overline{\mathfrak{B}}_n \right)^T$$

$$\mathbf{d}(x) := \left(0, 0, \ldots, 0, \underbrace{-\mathrm{e}^x}_{m^{\mathrm{th}}\mathrm{entry}}, 0, \ldots, 0, f_\gamma \beta h, -f_\gamma \beta h \right)^T$$

$$\hat{\kappa}(x) = \begin{pmatrix} 0 & s(x) & 0 & \cdots & & 0 \\ s(x) & 0 & s(x) & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & & 0 \\ 0 & \ldots & 0 & s(x) & & s(x) \\ 0 & \ldots & s(x) & \kappa(x) & & -\kappa_-(x) \\ 0 & \ldots & s(x) & -\kappa_+(x) & & \kappa(x) \end{pmatrix} \tag{11}$$

$$\kappa(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh \frac{\pi}{2} k \left(\frac{\pi}{\gamma} - (n+1) \right)}{2 \cosh \frac{\pi k}{2} \sinh \frac{\pi}{2} k \left(\frac{\pi}{\gamma} - n \right)} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k \tag{12}$$

$$\stackrel{\gamma \to 0}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{\pi}{2}|k|}}{2 \cosh \frac{\pi k}{2}} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k$$

$$s(x) = \frac{1}{2 \cosh x}, \quad f_\gamma = \frac{1}{2(1 - n\gamma/\pi)} \stackrel{\gamma \to 0}{=} \frac{1}{2}, \tag{13}$$

and the free energy contribution of the impurity is given by

$$f_i(T, h) = -T \begin{cases} \int_{-\infty}^{\infty} \frac{\ln Y_l(x)}{2\pi \cosh\left(x + \ln \frac{T}{T_K}\right)} \, \mathrm{d}x, \, l < m \\ \int_{-\infty}^{\infty} \frac{\ln[\mathfrak{B}_l \overline{\mathfrak{B}}_l](x)}{2\pi \cosh\left(x + \ln \frac{T}{T_K}\right)} \, \mathrm{d}x, \, l \geq m \end{cases}. \tag{14}$$

Again, the magnetic field is introduced by twisted boundary conditions. The integration kernel κ depends on n if $\gamma \neq 0$. The asymmetric driving term $-\mathrm{e}^x$ in the mth equation gives rise to different asymptotes in the limits $x \to \infty$, $x \to -\infty$, summarized in the following:

$$\ln Y_j^{(-\infty)} = 2 \ln \frac{\sinh \frac{\beta h}{2}(j+1)}{\sinh \frac{\beta h}{2}} \tag{15a}$$

$$\ln Y_{j<m}^{(\infty)} = 2 \ln \frac{\sin \frac{\pi}{m+2}(j+1)}{\sin \frac{\pi}{m+2}} \tag{15b}$$

$$\ln Y_{j \geq m}^{(\infty)} = 2 \ln \frac{\sinh \frac{\beta h}{2}(j - m + 1)}{\sinh \frac{\alpha \beta h}{2}} \tag{15c}$$

$$\ln \mathfrak{B}_n^{(-\infty)} = \ln \left[\mathrm{e}^{\frac{\beta h}{2} n} \frac{\sinh \frac{\beta h}{2}(n+1)}{\sinh \frac{\beta h}{2}} \right] \tag{15d}$$

$$\ln \mathfrak{B}_n^{(\infty)} = \ln \left[\mathrm{e}^{\frac{\alpha \beta h}{2}(n-m)} \frac{\sinh \frac{\alpha \beta h}{2}(n - m + 1)}{\sinh \frac{\alpha \beta h}{2}} \right]$$

with $\alpha = (1 - m\gamma/\pi)^{-1}$ and analogously for $\overline{\mathfrak{B}}_n^{(\pm\infty)} = \mathfrak{B}_n^{(\pm\infty)}\Big|_{h \to -h}$. The limits of $\ln y_j$, $\ln \mathfrak{b}_n$, $\ln \overline{\mathfrak{b}}_n$ are obtained from the above formulas by $y_j = Y_j - 1$, $\mathfrak{b}_n = \mathfrak{B}_n - 1$ and $\overline{\mathfrak{b}}_n = \overline{\mathfrak{B}}_n - 1$. For $\mathfrak{b}_n$, the results take the simple expression

$$\ln \mathfrak{b}_n^{(-\infty)} = \ln \left[\mathrm{e}^{\frac{\beta h}{2}(n+1)} \frac{\sinh \frac{\beta h}{2} n}{\sinh \frac{\beta h}{2}} \right]$$

$$\ln \mathfrak{b}_n^{(\infty)} = \ln \left[\mathrm{e}^{\frac{\alpha \beta h}{2}(n-m+1)} \frac{\sinh \frac{\alpha \beta h}{2}(n - m)}{\sinh \frac{\alpha \beta h}{2}} \right]$$

and similarly for $\overline{\mathfrak{b}}_n$ with $h \to -h$.

The above presented derivation of the closed set of finitely many integral equations for the impurity's free energy is speculative in the sense that an observation made for the well-understood case $S = 1/2$, $m = 1$ within a path integral approach [11] is extended to the case of general S and m. We intend to report on the generalization of [11] in a future publication. Our claim, however, that (10–13), (14) hold also in the general case follows by a simpler reasoning. By use of the algebra [15,17] of fused transfer matrices one can show that our key equations are equivalent to the infinitely many TBA equations of [6] in the isotropic case and to the equations of [8,9] in the anisotropic case. As an illustration, Figures 1–6 show the entropy and specific heat of the impurity for some special cases. The curves for $\gamma = 0$ agree nicely with similar graphs in [2,18]; those for $\gamma \neq 0$ are novel.
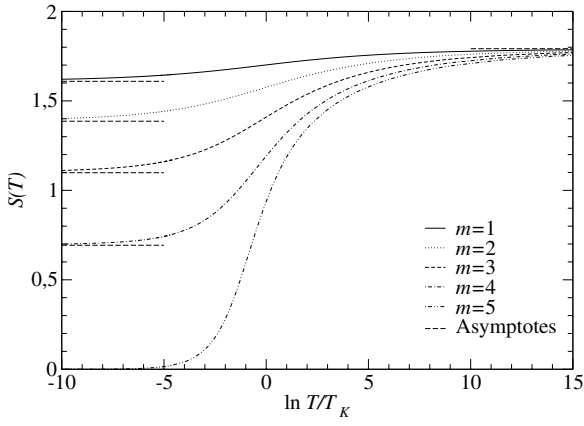
Fig. 1. Entropy for impurity spin $S = 5/2$ and different channel numbers. At low temperatures, the asymptotes $\ln(m+1)$ depend on the channel number, at high temperatures, the common asymptote $\ln(2S+1)$ is approached.
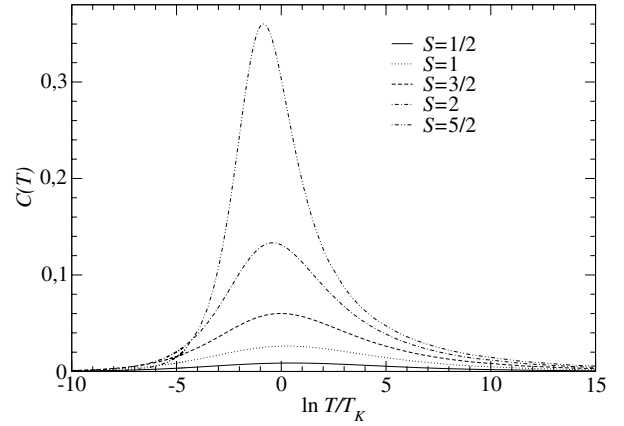

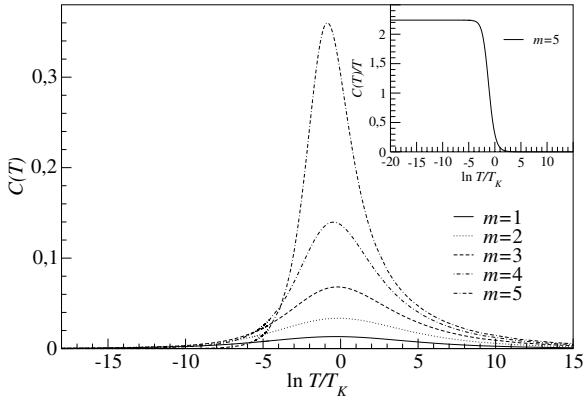**Fig. 2.** Specific heat for impurity spin $S = 5/2$ and different channel numbers. The inset shows the Fermi liquid behavior $\lim_{T \to 0} C(T)/T = \text{const.}$ for $2S = m = 5$.
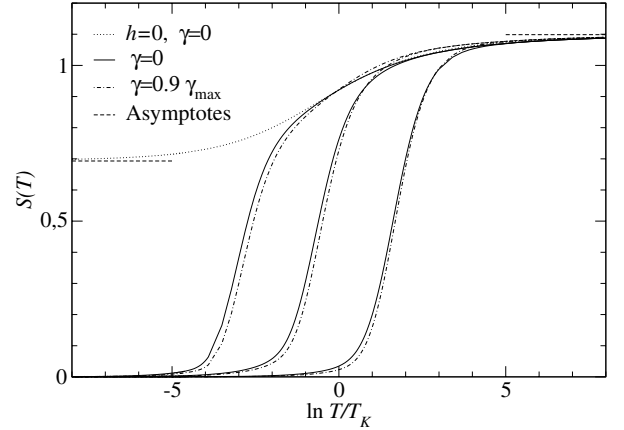

Fig. 3. Entropy for $m = 5$ channels and spins $S = 1/2, \ldots 5/2$. In the low temperature limit, the entropy takes the unusual value $\ln\{\sin[\pi(2S+1)/(m+2)]/\sin\pi/(m+2)\}$, equation (15b), whereas for high temperatures, the asymptotes are given by $\ln(2S+1)$.

Fig. 4. Specific heat for $m = 5$ channels and spins $S = 1/2, \ldots 5/2$.

Fig. 5. The entropy for the under-screened case $S = 1$, $m = 1$ with fields $h = 0, 0.1, 1, 10$ and anisotropies $\gamma = 0$, $0.9\gamma_{\max} = 0.707$ (for $\gamma_{\max}$, see Eq. (8)).

Fig. 6. The specific heat for the under-screened case $S = 1$, $m = 1$ with fields $h = 0, 0.1, 1, 10$ for anisotropies $\gamma = 0$, $0.9\gamma_{\max} = 0.707$ (for $\gamma_{\max}$, see Eq. (8)).
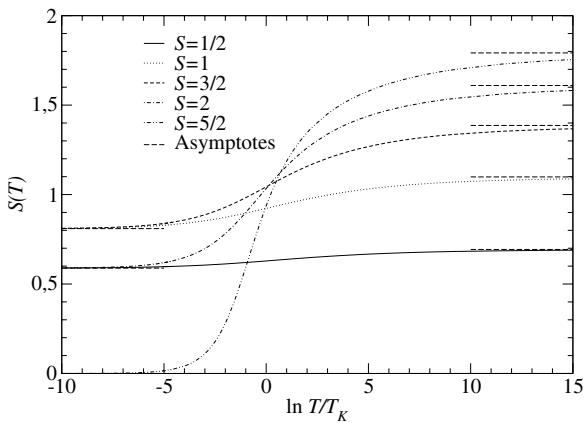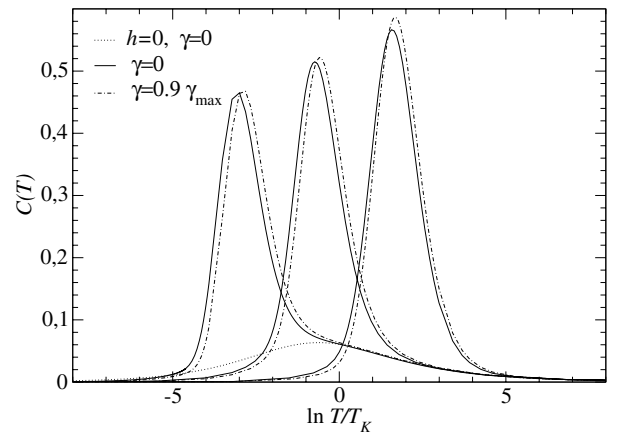
We expect that in the framework of a generalized lattice path integral approach, additional electron-electron interactions in the host must be introduced to keep the model integrable. These interactions may lead to both a renormalization of the Fermi velocity v_f and the electronic g-factor, analogous to the known exact solution [13]. These additional interactions will be the concern of future work; the analysis in this work is based on the model (2a–2c).

3 Low temperature evaluation

3.1 2S = m: Exact screening

The first non-vanishing order in a low temperature, low field expansion of the free energy in the exactly screened case can be obtained by expanding the corresponding integral in the following form

$$f_i(T,h) = -\frac{T}{2\pi} \int_{-\infty}^{\infty} \frac{\ln\left[\mathfrak{B}_l \overline{\mathfrak{B}}_l\right](x)}{\cosh\left(x + \ln\frac{T}{T_K}\right)} \mathrm{d}x$$
$$\overset{T,h \ll T_K}{=} -\frac{T^2}{\pi T_K} \int_{-\infty}^{\infty} \mathrm{e}^x \ln\left[\mathfrak{B}_l \overline{\mathfrak{B}}_l\right](x)\, \mathrm{d}x. \quad (16)$$

The temperature and the field are supposed to be small compared to T_K. As can be seen from equations (15), the integral in equation (16) only exists for $l = m$. The above approximation makes sense if $T, h \ll T_K$. The integral in equation (16) can be done exactly, by a generalization of a method used in [19] leading to dilogarithms and applied for instance to the spin-S Heisenberg chain in [17,20].

In the notation of equation (10), consider the integral

$$I(T,h) := \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} \mathbf{y}(x)\, \mathbf{Y}(x) - \mathbf{y}(x) \frac{\mathrm{d}}{\mathrm{d}x} \mathbf{Y}(x) \right) \mathrm{d}x. \quad (17)$$

From the symmetry of the integration kernels one deduces

$$f_i(T,h) = \frac{T^2}{2\pi T_K} I(T,h) - \frac{f_\gamma}{2\pi T_K} h^2 m. \quad (18)$$

On the other hand, the integral $I(T,h)$ is calculated as in [17] by using dilogarithmic identities, the result is

$$I = -\pi^2 \frac{m}{m+2}. \quad (19)$$

Combining equations (18, 19), one gets the first term in an expansion of the free energy for low fields and temperatures:

$$f_i(T,h) = -\frac{T^2}{2T_K} \frac{\pi m}{m+2} - \frac{m}{4(\pi - m\gamma)} \frac{h^2}{T_K}. \quad (20)$$

From equation (20),

$$C_i(T) = \frac{T}{T_K} \frac{\pi m}{m+2} \qquad \chi_i(h) = \frac{1}{T_K} \frac{m}{2(\pi - m\gamma)}. \quad (21)$$

These relations exhibit clearly the role of T_K as a "low temperature scale". According to equation (21), the impurity contribution to the specific heat and magnetic susceptibility are Fermi liquid like at $T, h \ll T_K$ in the exactly screened case $l = m$. This is the regime of "strong coupling": The anti-ferromagnetic spin exchange leads to the formation of a many particle state between the impurity and the host electrons, which screens the magnetic moment of the impurity. Elementary excitations of this bound state are Fermi like. Nozières [1,21] built up a phenomenological Fermi liquid theory to describe this regime. This Fermi liquid behavior is to be compared with the host. It consists of spin-1/2-fermions of m non-interacting channels (or of m flavors), so that the density of states is enhanced by a factor of m

$$C_h(T) = T \frac{m\pi^2}{3} \rho_h \qquad \chi_h(h) = \frac{m\,\rho_h}{4}. \quad (22)$$

The coefficient of the linear T-dependence of C_i (C_h) is denoted by δ_i (δ_h). The low-temperature Wilson ratio R is defined and calculated as

$$R := \lim_{T \to 0} \frac{\chi_i}{\chi_h} \frac{\delta_h}{\delta_i} = \frac{2(m+2)}{3\left(1 - m\frac{\gamma}{\pi}\right)} \geq 2. \quad (23)$$

The lower bound 2 is reached for $m = 1$, $\gamma = 0$. The striking feature in comparing equations (21, 22) is that C_i is reduced by a factor of $3/(2m+4)$ in comparison to C_h, if the constant ρ_h is chosen such that $\chi_i = \chi_h$ for $\gamma = 0$. This may be interpreted by the localization of the impurity: Contrary to the host electrons, it does not move, so that the specific heat is reduced.

3.2 Linearization

In this section, the ground-state and the lowest T-dependent contribution to the free energy are calculated by linearizing the NLIE for $h \neq 0$. From equations (15) one observes that for $h \neq 0$ and $\beta \to \infty$ in the limit $x \to -\infty$ the auxiliary functions scale with βh. Especially, $\lim_{\beta \gg 1} \ln \overline{\mathfrak{B}}_l = 0 + \mathcal{O}\left(\mathrm{e}^{-\beta h}\right)$, so that one can neglect $\ln \overline{\mathfrak{B}}_l$ with exponential accuracy.

One introduces the scaling functions ϵ_j,

$$\ln y_j(x) =: \beta h K_m\, \epsilon_j(-x + \ln \beta h K_m)\ \ j = 1, \ldots, l-1$$
$$\ln \mathfrak{b}_l(x) =: \beta h K_m\, \epsilon_l(-x + \ln \beta h K_m),$$

where K_m is defined by $\ln y_m(\ln \beta h K_m) = 0$. The shift in the spectral parameter is performed in order to deal with functions which have a zero in the origin,

$$\epsilon_m(0) = 0. \quad (24)$$

Note that therefore

$$\epsilon_j(x) \begin{cases} > 0, \ x > \ln K_m\,;\ j = m \\ < 0, \ x < \ln K_m\,;\ j = m \\ > 0\,; \qquad\qquad\quad \text{otherwise,} \end{cases} \quad (25)$$

so that one linearizes the logarithms

$$\ln\left[1 + \exp\left(\beta h K_m \epsilon_j(x)\right)\right] =$$

$$\begin{cases} \beta h K_m \epsilon_j \theta(x) + \mathcal{O}(\text{const.}), & j < m \\ \beta h K_m \epsilon_m \theta(x) + \frac{\pi^2}{6\beta h K_m|\epsilon_m(0)|}\,\delta(x), & j = m \\ \beta h K_m \epsilon_j + \mathcal{O}(\exp(-\text{const.}\beta h)) & j > m. \end{cases} \quad (26)$$

The linearization is done with exponential accuracy for $\epsilon_{j\geq m}$, but with only algebraic accuracy for $\epsilon_{j<m}$ because of the constant asymptotes of the latter functions. The first T-contribution for these is calculated below within a different linearization scheme. Here, the lowest T-contribution for the exactly and under-screened single channel case $l \geq m = 1$ is determined. For $\gamma = 0$, the results of [13] are confirmed.

It is convenient to define a matrix $\hat{A}^{-1}$

$$\hat{A}_{i,j}^{-1}(k) := \begin{cases} 1 - \mathcal{F}_k[\kappa] & i = j = n \\ \delta_{i,j} - \mathcal{F}_k[s](\delta_{i,j+1} + \delta_{i,j-1}) & \text{otherwise.} \end{cases} \quad (27)$$

By inserting the ϵ-functions into the original set of NLIE and using equations (26, 27) one obtains

$$\left[\hat{A}_{ij}^{-1} * \epsilon_j\right] = \delta_{i,m}\left(\tilde{d}_m - \epsilon_m^-\right) + \delta_{i,l}\frac{f_\gamma}{K_m}, \quad (28)$$

where we have defined the driving term

$$\tilde{d}_m(x) := -\mathrm{e}^{-x} + \frac{\pi^2}{6\beta h K_m|\epsilon_m'(0)|}\,\delta(x),$$

and

$$\epsilon_j(x) = \begin{cases} \epsilon_{j,0}^+(x), \ j < m \\ \epsilon_{j,0}^+(x) + \frac{\pi^2}{6(\beta h K_m)^2|\epsilon_m'(0)|}\left(U_j^+(x) + \delta(x)\right), \\ \qquad\qquad j = m \\ \epsilon_{j,0}(x) + \frac{\pi^2}{6(\beta h K_m)^2|\epsilon_m'(0)|}\,U_j(x), \ j > m. \end{cases} \quad (29)$$

The terms $\epsilon_{j,0}$ contain the ground-state, and from the rest, the lowest T-dependent contributions are obtained. We will calculate the latter explicitly for $l \geq m = 1$. We made use of the shorthand-notations

$$\epsilon_j(x)\,\theta(x) =: \epsilon_j^+(x), \qquad \epsilon_j(x)\,\theta(-x) =: \epsilon_j^-(x)$$

for functions in direct space. Their Fourier transforms are denoted as

$$\mathcal{F}_k\left[\epsilon_j^+\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\epsilon_j^+(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x =: \epsilon_j^+(k),$$

hence an index $^+$ denotes analyticity in the upper half of the complex k-plane. Define a new energy scale

$$T_h := \frac{T_K}{K_m}. \quad (30)$$

Then the free energy, magnetization and specific heat are given by

$$f_i(h) = -\frac{hK_m}{2}\int_{-\infty}^{\infty}\frac{\epsilon_l(k)}{\cosh\frac{\pi k}{2}}\mathrm{e}^{-\mathrm{i}k\ln\frac{h}{T_h}}\,\mathrm{d}k \quad (31)$$

$$M_i(h) = \frac{K_m}{2}\int_{-\infty}^{\infty}\frac{1-\mathrm{i}k}{\cosh\frac{\pi k}{2}}\epsilon_{l,0}(k)\mathrm{e}^{-\mathrm{i}k\ln\frac{h}{T_h}}\,\mathrm{d}k \quad (32)$$

$$C_i(T,h) = \frac{\pi^2 T}{6hK_m|\epsilon_m'(0)|}\int_{-\infty}^{\infty}\frac{U_l(k)\mathrm{e}^{-\mathrm{i}k\ln\frac{h}{T_h}}}{\cosh\frac{\pi k}{2}}\,\mathrm{d}k. \quad (33)$$

Obviously, the Wilson ratio $C_i/(T\chi_i)$ is a function of h alone. In the following, the magnetization and the susceptibility are calculated from $\epsilon_{l,0}$, and the specific heat from U_l, defined in equation (29).

3.3 Under-screened and exactly screened cases, 2S ≥ m

The linear system (28) is solved by inverting the matrix $\hat{A}^{-1}(k)$. This gives the matrix $\hat{A}(k)$ with elements

$$\left[\hat{A}(k)\right]_{i,j} = \frac{1}{\det\hat{A}^{-1}(k)}\,\alpha_{i,j}(k), \quad (34)$$

where $\alpha_{i,j}$ is the adjunct to $\left[\hat{A}^{-1}\right]_{i,j}$ in $\det\hat{A}^{-1}$. Both quantities are seen to satisfy recursion relations, which can be solved explicitly with the results

$$\det\hat{A}^{-1}(k) = \frac{\sinh\frac{\pi}{\gamma}\frac{\pi k}{2}}{\left(2\cosh\frac{\pi k}{2}\right)^l \sinh\left(\frac{\pi}{\gamma} - l\right)\frac{\pi k}{2}}$$

$$\hat{A}_{j,j}(k) = \frac{2\cosh\frac{\pi k}{2}\sinh j\frac{\pi k}{2}\sinh\left(\frac{\pi}{\gamma} - j\right)\frac{\pi k}{2}}{\sinh\frac{\pi k}{2}\sinh\frac{\pi}{\gamma}\frac{\pi k}{2}}$$

$$\hat{A}_{j,l}(k) = \frac{2\cosh\frac{\pi k}{2}\sinh j\frac{\pi k}{2}\sinh\left(\frac{\pi}{\gamma} - l\right)\frac{\pi k}{2}}{\sinh\frac{\pi k}{2}\sinh\frac{\pi}{\gamma}\frac{\pi k}{2}}. \quad (35)$$

Other matrix elements are not needed. Then

$$\frac{\epsilon_m(k)}{\hat{A}_{m,m}(k)} = -\epsilon_m^-(k) + \tilde{d}_m(k) + \frac{\hat{A}_{m,l}(k)}{\hat{A}_{m,m}(k)}\frac{f_\gamma}{K_m}\delta(k) \quad (36)$$

$$\epsilon_l(k) = \frac{\hat{A}_{m,l}(k)}{\hat{A}_{m,m}(k)}\epsilon_m(k)$$

$$+ \left(\hat{A}_{l,l}(k) - \frac{\hat{A}_{m,l}^2(k)}{\hat{A}_{m,m}(k)}\right)\frac{g}{K_m}\delta(k). \quad (37)$$

Equation (36) is solved by factorizing $\hat{A}_{m,m}$ into two functions $G_\pm$, G_+ (G_-) being analytic in the upper (lower) half of the complex k-plane (Wiener-Hopf factorization),

$$\hat{A}_{m,m}(k) = G_+(k)\,G_-(k). \quad (38)$$

$$M_i(h) = \frac{l-m}{2} \frac{1}{1-m\frac{\gamma}{\pi}}$$

$$+ \frac{m}{4\pi^{3/2}} \frac{1-\frac{\gamma}{\pi}l}{1-\frac{\gamma}{\pi}m} \int_{-\infty}^{\infty} \mathrm{i} \frac{\Gamma\left(\frac{1}{2}+\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{\pi k}{2\gamma}\right) \Gamma\left(1+\mathrm{i}\frac{k}{2}\left(\frac{\pi}{\gamma}-m\right)\right)}{\Gamma\left(1+\mathrm{i}\frac{k}{2}\left(\frac{\pi}{\gamma}-l\right)\right) \Gamma\left(1-\mathrm{i}\frac{k}{2}\left(\frac{\pi}{\gamma}-l\right)\right) \Gamma\left(1-\mathrm{i}\frac{mk}{2}\right)} \frac{\mathrm{e}^{-\mathrm{i}k\ln\frac{h}{T_h}-\mathrm{i}ak}}{k+\mathrm{i}0^+} \mathrm{d}k. \quad (42)$$

The result is

$$\epsilon_{m,0}^+(k) = \frac{1}{2\pi} \frac{G_+(k)\,G_-(-\mathrm{i})}{(k+\mathrm{i}0^+)(k+\mathrm{i})} \quad (39)$$

$$U_1^+(k)\big|_{m=1} + 1 = \frac{1}{2\pi}\,G_+(k),\, l > 1, \quad (40)$$

and

$$G_+(k) = \left(\frac{2\pi m\left(1-\frac{\gamma}{\pi}l\right)}{1-\frac{\gamma}{\pi}(l-m)}\right)^{\frac{1}{2}}$$

$$\times \frac{\Gamma\left(1-\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{k}{2}\frac{\pi}{\gamma}\right) \mathrm{e}^{-\mathrm{i}ak}}{\Gamma\left(\frac{1}{2}-\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}m\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{k}{2}\left(\frac{\pi}{\gamma}-m\right)\right)}$$

$$G_-(k) = G_+(-k).$$

The constant a is chosen to be $a = -\frac{\pi}{2\gamma}\ln(1-\gamma m/\pi) + \frac{m}{2}\ln(1-\pi/(\gamma m))$. From equation (36) the constant K_m is found to be

$$K_m = f_\gamma \frac{\hat{A}_{m,l}(0)}{\hat{A}_{m,m}(0)} \frac{G_-(0)}{G_-(-\mathrm{i})}. \quad (41)$$

With this information and equations (6, 30) we know the magnetic field scale T_h whose role is revealed in various expansions of the magnetization $m(h)$ below.

Ground state

The ground state contribution to the free energy equation (31) is given by inserting $\epsilon_{l,0}$ from equation (37) with equation (39).

The magnetization is written down from equation (32),

See equation (42) above.

The integral can be calculated by closing the contour in the lower ($h > T_h$) or in the upper ($h < T_h$) half plane. This results in two power series in T_h/h for $h > T_h$ and in h/T_h for $h < T_h$ with integer and non-integer powers, depending on the poles of the integrand. These calculations are straightforward but lengthy. Especially, in the limits

$h \ll T_h$, $h \gg T_h$, one extracts the constant values

$$M_i(h) \to \begin{cases} \frac{l-m}{2}\frac{1}{1-m\frac{\gamma}{\pi}} + \mathcal{O}\left(\left(\frac{h}{T_h}\right)^{2\gamma/(\pi-m\gamma)}\right); \\ \qquad l > m, h \ll T_h \\ \mathcal{O}\left(\frac{h}{T_h}\right); l = m, h \ll T_h \\ \frac{l}{2} + \mathcal{O}\left(\left(\frac{h}{T_h}\right)^{-2\gamma/\pi}\right); \quad h \gg T_h. \end{cases} \quad (43)$$

Thus for $h = 0$, a non-integer rest-spin remains in the case $\gamma \neq 0$, $l > m$ and h-dependent corrections are of non-integer powers of h. The constant terms in (43) agree precisely with the asymptotes, equations (15) and with [8,9], where the same model was investigated by TBA-techniques in the limits indicated in (43).

In the isotropic limit $\gamma \to 0$

$$G_+(k) = \frac{\sqrt{2\pi m}\,\Gamma\left(1-\mathrm{i}\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}-\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{km}{2}\right)} \left(\frac{-\mathrm{i}km}{2\mathrm{e}}\right)^{-\mathrm{i}\frac{mk}{2}}$$

$$M_i(h) = \frac{m}{4\pi^{3/2}} \int_{-\infty}^{\infty} \mathrm{i} \frac{\Gamma\left(\frac{1}{2}+\mathrm{i}\frac{k}{2}\right) \Gamma\left(1-\mathrm{i}\frac{k}{2}\right)}{\Gamma\left(1-\mathrm{i}\frac{mk}{2}\right)}$$

$$\times \left(-\frac{\mathrm{i}km}{2\mathrm{e}}\right)^{-\mathrm{i}\frac{km}{2}} \mathrm{e}^{-\frac{\pi|k|}{2}(l-m)} \frac{\mathrm{e}^{-\mathrm{i}k\ln\frac{h}{T_h}}}{k+\mathrm{i}0^+} \mathrm{d}k$$

$$+ \frac{l-m}{2} \to \begin{cases} \frac{l-m}{2}, & h \ll T_h \\ \frac{l}{2}, & h \gg T_h \end{cases}. \quad (44)$$

In the last line, only the leading behavior due to the simple pole at $-\mathrm{i}0^+$ for high fields has been included. The integral in equation (44) has been given by Tsvelick and Wiegmann, [4]. It allows to determine the zero-temperature scales defined below. There are several singularities of the integrand equation (44) in the complex k-plane:

i) $m = l$: Poles are distributed in the upper and lower half-planes, with an additional dominating cut along the negative imaginary axis.
ii) $m < l$: A cut along the whole imaginary axis goes along with sub-leading poles in both the upper and lower half planes.

Singularities in the lower (upper) half plane are relevant for $h > T_h$ ($h < T_h$). Poles $k_n = \mathrm{i}(2n+1)$, $n = 0, 1, \ldots$ in the upper half plane only give a leading contribution in the exactly screened case $l = m$. They have residuals

$2\mathrm{i}(-1)^{n+1}/n!$, such that the magnetization is given by a series

$$M_i(h < T_h) \stackrel{m=l}{=} \frac{m}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(m\left(n+\frac{1}{2}\right)\right)}$$
$$\times \left(\frac{m\left(n+\frac{1}{2}\right)}{\mathrm{e}}\right)^{m\left(n+\frac{1}{2}\right)} \left(\frac{h}{T_h}\right)^{2n+1}.$$

One recognizes the signature of a Fermi liquid in first order: $M(h) \propto h$ and $\chi(h = 0) = $const. Upon inserting explicit values, one finds agreement with equation (21). We shall establish this agreement explicitly for arbitrary γ below, equations (55, 56).

Let us draw our attention to the cut in the lower half plane, for values $l \geq m$. By linearizing the integrand, we find

$$M_i(h > T_h) - \frac{l}{2} \stackrel{l \geq m}{=} -\frac{ml}{4} \left(\frac{1}{\ln \frac{h}{T_h}} + \frac{m}{2} \frac{\ln \ln \frac{h}{T_h}}{\ln^2 \frac{h}{T_h}} \right.$$
$$\left. + \frac{A}{\ln^2 \frac{h}{T_h}} + \mathcal{O}\left(\ln^{-3} h/T_h\right) \right)$$
$$A = \frac{1}{2}\left[-m\ln m + (m-2)\ln 2\right]. \quad (45)$$

The contribution $\mathcal{O}(\ln^{-2} h/T_h)$ is absorbed by the definition

$$T_h = \mathrm{e}^{-A}\widetilde{T}_h \quad (46)$$

$$M_i(h > \widetilde{T}_h) - \frac{l}{2} \stackrel{l \geq m}{=} -\frac{ml}{4} \left(\frac{1}{\ln \frac{h}{\widetilde{T}_h}} + \frac{m}{2} \frac{\ln \ln \frac{h}{\widetilde{T}_h}}{\ln^2 \frac{h}{\widetilde{T}_h}} \right.$$
$$\left. + \mathcal{O}\left(\ln^{-3} h/\widetilde{T}_h\right) \right). \quad (47)$$

Equation (46) defines the zero-temperature scale for large magnetic fields.

Finally, upon "encircling" the cut in the upper half plane, one only replaces the pre-factor l in equation (47) by $l - m$ and sets $-\ln h/\widetilde{T}_h = \ln \widetilde{T}_h/h$:

$$M_i(h < \widetilde{T}_h) \stackrel{l \geq m}{=} \frac{l-m}{2} + \frac{m(l-m)}{4} \left(\frac{1}{\ln \frac{\widetilde{T}_h}{h}} \right.$$
$$\left. - \frac{m}{2} \frac{\ln \ln \frac{\widetilde{T}_h}{h}}{\ln^2 \frac{\widetilde{T}_h}{h}} + \mathcal{O}\left(\ln^{-3} \widetilde{T}_h/h\right) \right). \quad (48)$$

Note that this simple replacement does not hold for $\gamma \neq 0$, as can already be seen from the lowest order, equation (43).

The free spin value of the magnetization is approached logarithmically at high fields. This asymptotical freedom in the weak coupling limit is a genuine feature of the Kondo model. An analogous effect occurs for low fields in the under-screened case; however, the first correction is of

opposite sign compared to the high-temperature case, cf. equations (47, 48). Classical Fermi liquid behavior appears at low temperatures if the impurity is exactly screened. The physical origin of these results has been revealed by Nozières [1] already before the exact solution of the Kondo model was known: At high fields, corrections to the asymptotical freedom of the impurity spin are caused by the weak anti-ferromagnetic coupling with the host particles. At low fields, the impurity spin is partially screened due to strong anti-ferromagnetic exchange. Two kinds of interactions with this impurity-electron system may occur: On the one hand, a weak *ferro*magnetic coupling of the residual spin $S - m/2$ with the host, due to the Pauli principle (this explains the change of sign in the leading order on the rhs of Eqs. (47, 48)). On the other hand, a polarization of the bound complex by host electrons, analogously to the Fermi liquid excitations at $S = m/2$. This polarization is given by the poles and dominated by the ferromagnetic Kondo interactions, reflected by the cuts in the complex plane.

Finite temperature

After the calculation of the magnetization, we proceed with the specific heat for $l \geq m = 1$. By inserting equation (40) into equation (33), we get

$$C_i(T \ll T_K, h)/T = \frac{\pi^2}{3hK_1|\epsilon'_1(0)|} V(-\ln h/T_h),$$

where the Fourier transform of $V(x)$ is given by

$$V(k) = \begin{cases} s(k)\left[U_1^+(k) + 1\right], & l = 1 \\ \frac{A_{1l}(k)}{A_{11}(k)} s(k)\left[U_1^+(k) + 1\right], & l \geq 1 \end{cases}.$$

From equation (40) it follows for $l = 1$ that $V(k) = G_+(k)s(k)$, so that

$$C_i(T \ll T_K, h)/T = \frac{\pi^2}{3|\epsilon'_1(0)|G_-(-\mathrm{i})K_1^2} \chi(h), \quad (49)$$

where we have inserted equations (32, 39, 40). The constant $|\epsilon'_1(0)|$ is determined from the condition that equation (49) matches equation (21) for $l = 1 = m$, so that $|\epsilon'_1(0)| = 1/(K_1 G_-(0))$, and

$$C_i(T \ll T_K, h)/T = \frac{2\pi(\pi - l\gamma)}{3} \chi_i(h). \quad (50)$$

3.4 Over-screened and exactly screened cases, $2S \leq m$ (ground state)

In this section, we calculate the ground state for $l \leq m$, finite temperatures are dealt with in the next section.

If $l \leq m$, there are m equations, the last one being

$$\epsilon_m(x) = -\mathrm{e}^{-x} + \frac{f_\gamma}{K_m} + [s * \epsilon_{m-1} + \kappa * \epsilon_m](x),$$

$$M_i(h) = \frac{l}{4\pi^{3/2}} \int_{-\infty}^{\infty} i \frac{\Gamma\left(\frac{1}{2}+i\frac{k}{2}\right) \Gamma\left(1+im\frac{k}{2}\right) \Gamma\left(1-i\frac{k}{2}\right) \Gamma\left(1-i\frac{k}{2}\frac{\pi}{\gamma}\right)}{\Gamma\left(1-i\frac{k}{2}\left(\frac{\pi}{\gamma}-m\right)\right) \Gamma\left(1-il\frac{k}{2}\right) \Gamma\left(1+il\frac{k}{2}\right)} \frac{e^{-ik\ln\frac{h}{T_h}-iak}}{k+i0^+} dk \overset{h\gg T_h}{\to} \frac{l}{2}. \tag{53}$$

where $g = \frac{1}{2(1-m\gamma/\pi)}$. On the right-hand side of equation (28), only the mth entry is different from zero. Consequently,

$$\frac{\epsilon_m^+(k)}{\hat{A}_{m,m}(k)} = -\epsilon_m^-(k) + \tilde{d}_m(k) + \frac{f_\gamma}{K_m}\delta(k) \tag{51}$$

$$\epsilon_l = \left[\frac{\hat{A}_{l,m}}{\hat{A}_{m,m}}\right]\epsilon_m^+. \tag{52}$$

Equations (36, 51) are solved by the Wiener-Hopf method, afterwards ϵ_m^+ is inserted in equations (37, 52). The relevant matrix entries read

$$\hat{A}_{m,m}(k) = \frac{2\cosh\frac{\pi}{2}k \sinh m\frac{\pi k}{2} \sinh\frac{\pi k}{2}\left(\frac{\pi}{\gamma}-m\right)}{\sinh\frac{\pi k}{2} \sinh\frac{\pi}{\gamma}\frac{\pi k}{2}}$$

$$\hat{A}_{l,m}(k) = \frac{2\cosh\frac{\pi}{2}k \sinh l\frac{\pi k}{2} \sinh\frac{\pi k}{2}\left(\frac{\pi}{\gamma}-m\right)}{\sinh\frac{\pi k}{2} \sinh\frac{\pi k}{2}\frac{\pi}{\gamma}}.$$

With $\hat{A}_{m,m}(k) = G_+(k)G_-(k)$, the auxiliary functions are given by

$$\epsilon_m^+(k) = \frac{1}{2\pi} \frac{G_+(k)G_-(-i)}{(k+i0^+)(k+i)}$$

$$G_+(k) = \frac{\left(2m\pi\left(1-m\frac{\gamma}{\pi}\right)\right)^{\frac{1}{2}} \Gamma\left(1-i\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}-i\frac{k}{2}\right) \Gamma\left(1-im\frac{k}{2}\right)}$$

$$\times \frac{\Gamma\left(1-i\frac{\pi}{\gamma}\frac{k}{2}\right) e^{-iak}}{\Gamma\left(1-i\frac{k}{2}\left(\frac{\pi}{\gamma}-m\right)\right)},$$

where $K_m = f_\gamma \frac{G_-(0)}{G_-(-i)}$. In the limit of high fields $h \gg T_h$, the magnetization reads

See equation (53) above.

The isotropic limit $\gamma \to 0$ yields

$$G_+(k) = \frac{\sqrt{2m\pi} \Gamma\left(1-i\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}-i\frac{k}{2}\right) \Gamma\left(1-im\frac{k}{2}\right)} \left(-\frac{ikm}{2e}\right)^{-i\frac{mk}{2}}$$

$$M_i(h) = \frac{l}{4\pi^{3/2}} \int_{-\infty}^{\infty} i \frac{\Gamma\left(\frac{1}{2}+i\frac{k}{2}\right) \Gamma\left(1+im\frac{k}{2}\right) \Gamma\left(1-i\frac{k}{2}\right)}{\Gamma\left(1-il\frac{k}{2}\right) \Gamma\left(1+il\frac{k}{2}\right)}$$

$$\times \left(-\frac{ikm}{2e}\right)^{-i\frac{mk}{2}} \frac{e^{-ik\ln\frac{h}{T_h}}}{k+i0^+} dk.$$

Note that for $\gamma = 0$, a cut along the negative part of the imaginary axis occurs. It dominates the poles in the lower half of the complex plane, and one is faced with the

expected Kondo behavior for $h > T_h$, again in accordance with [4].

For $h < T_h$, the leading behavior is given by the poles with smallest positive imaginary part.

$$M_i(h) \propto \left(\frac{h}{T_K}\right)^{2/m}, \quad \chi(h) \propto \left(\frac{h}{T_K}\right)^{2/m-1}. \tag{54}$$

Thus over-screening induces non-integer exponents of h, independent of γ, at $T = 0$.

Finally note that for $l = m$, the two expressions equations (44, 53) coincide. It is shown that the first non-vanishing order linear in h of the magnetization leads to the $T = 0$, $h = 0$ susceptibility, calculated in equation (21). For $l = m$, the magnetization reads

$$M_i(h) = \frac{m}{4\pi^{3/2}} \int_{-\infty}^{\infty} i \frac{\Gamma\left(\frac{1}{2}+i\frac{k}{2}\right) \Gamma\left(1-i\frac{k}{2}\right) \Gamma\left(1-i\frac{\pi k}{2\gamma}\right)}{\Gamma\left(1-i\frac{mk}{2}\right) \Gamma\left(1-i\frac{k}{2}\left(\frac{\pi}{\gamma}-m\right)\right)}$$

$$\times \frac{e^{-ik\ln\frac{h}{T_h}-iak}}{k+i0^+} dk.$$

Being interested in fields $h < T_h$, one takes account of the poles at $k_n := i(2n+1)$ with residuals $2i(-1)^{n+1}/n!$. These result in the series

$$M_i(h) = \frac{1}{\pi^{1/2}\left(1-\frac{\gamma}{\pi}m\right)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

$$\times \frac{\Gamma\left(\frac{1}{2}+n\right) \Gamma\left(\frac{\pi}{\gamma}\left(n+\frac{1}{2}\right)\right) a^{2n+1}}{\Gamma\left(m\left(\frac{1}{2}+n\right)\right) \Gamma\left(\left(\frac{\pi}{\gamma}-m\right)\left(n+\frac{1}{2}\right)\right)} \left(\frac{h}{T_h}\right)^{2n+1}.$$

From the definition of T_h, equation (30), one gets to first order in h:

$$\lim_{h\ll T_h} M_i(h) = \frac{m}{2\pi\left(1-\frac{\gamma}{\pi}m\right)} \frac{h}{T_K} \tag{55}$$

$$\chi_i(T=0,h=0) = \frac{m}{2\pi\left(1-\frac{\gamma}{\pi}m\right)} \frac{1}{T_K}. \tag{56}$$

This result is expected from equation (21).

Finite temperature

As pointed out above, we are not able to account for corrections of the linearized functions $\epsilon_{j<m}$, $m > 1$ in the framework of the rigorous linearization. In the following, we consider the case $j < m$ for small magnetic fields and low temperatures. For $\gamma = 0$, this case was treated

previously by Affleck [22] using bosonization and CFT-techniques. He calculated the low-temperature Wilson-ratio analytically for arbitrary m. Analogous results have been obtained by the TBA-solution [23] only for $m = 2$, $S = 1/2$. Here, we will confirm Affleck's findings for $m \geq 2$, extended to the anisotropic case $\gamma \neq 0$.

First consider the case $m = 2$, $2S = 1$. The relevant auxiliary functions are given by:

$$\ln y_1(x) = \left[s * \ln \mathfrak{B}_2 \overline{\mathfrak{B}}_2 \right](x)$$

$$\ln \mathfrak{b}_2(x) = -e^x + \frac{\beta h}{2} + [s * \ln Y_1](x) + \left[\kappa * \ln \mathfrak{B}_2 \right](x)$$
$$- \left[\kappa_- * \ln \overline{\mathfrak{B}}_2 \right](x)$$

$$f_i(T, h) = -\frac{T}{2\pi} \int_{-\infty}^{\infty} \frac{\ln Y_1(x)}{\cosh\left(x + \ln \frac{T}{T_K}\right)} \, dx. \qquad (57)$$

In Section 3.1, we found (Eq. (20) specialized to $m = 2$):

$$\lim_{x \gg 0} [s * \ln \mathfrak{B}_2 \overline{\mathfrak{B}}_2](x) = e^{-x} \left(\frac{\pi}{4} - \frac{(\beta h)^2}{2(\pi - 2\gamma)} \right).$$

Consequently,

$$\lim_{x \gg 0} \ln Y_1(x) = \ln 2 + e^{-x} \left(\frac{\pi}{8} - \frac{(\beta h)^2}{4(\pi - 2\gamma)} \right) \qquad (58)$$

One approximates the kernel in the same scheme as in equation (20),

$$\lim_{T \ll T_K} \frac{1}{\cosh\left(x + \ln T/T_K\right)} = \frac{2T}{T_K} e^x, \ x \leq \ln \frac{T}{T_K}. \qquad (59)$$

The approximations (58, 59) lead to a constant integrand, which we deal with by introducing a cutoff D,

$$f_i(T, h) = -\frac{T}{4\pi} \ln 2$$
$$- \frac{T^2}{T_K} \left[\frac{1}{8} + \frac{(\beta h)^2}{4\pi(\pi - 2\gamma)} \right] \int_{D}^{\ln T/T_K} 1 \, dx, \ D \ll 0$$

The cutoff D depends neither on T nor on h, so the occurrence of the logarithm $\ln T/T_K$ in the above equation leads to

$$\lim_{T, h \ll T_K} C_i(T, h) = -\frac{1}{4} \frac{T \ln T/T_K}{T_K} \qquad (60a)$$

$$\lim_{T, h \ll T_K} \chi_i(T, h) = -\frac{1}{2\pi(\pi - 2\gamma)} \frac{\ln T/T_K}{T_K} \qquad (60b)$$

$$R_w|_{m=2, 2S=1} = \frac{8}{3(1 - 2\gamma/\pi)}, \qquad (60c)$$

confirming, for $\gamma = 0$, numerical findings by Sacramento [23] and Affleck's prediction.

For values $m > 2$, we perform an asymptotic linearization of the NLIE in the region $x \to \infty$, following [4]. Corrections to $\ln Y_j^{(\infty)}$ are expressed through a correction function $D_j(x)$,

$$\ln Y_j(x) = \ln Y_j^{(\infty)} + D_j(x), \lim_{x \to \infty} D_j(x) = 0.$$

Linearizing $\ln y_j$ to first order in D_j, $j = 1, \ldots, m - 1$

$$\ln y_j(x) = \frac{1}{2} \ln Y_{j-1}^{(\infty)} Y_{j+1}^{(\infty)} + \frac{f_j^2}{f_{j-1} f_{j+1}} D_j(x)$$

$$f_j := \left(Y_j^{(\infty)} \right)^{1/2} = \frac{\sin \frac{\pi}{m+2}(j+1)}{\sin \frac{\pi}{m+2}}$$

$$D_j(x) = \frac{f_{j-1} f_{j+1}}{f_j^2} \left[s * (D_{j-1} - D_{j+1}) \right](x) \qquad (61)$$

$$D_m \equiv \ln \mathfrak{B}_m \overline{\mathfrak{B}}_m. \qquad (62)$$

Equation (61) has been derived for $x \gg 0$. This is approximately accounted for by writing $D_j(x) \equiv D_j(x) \, \theta(x)$. The linearized equations form an algebraic system by Fourier transforming. Each unknown function $D_j(x)$ can be expressed in terms of D_m,

$$D_j(x) = [t_j * D_m](x) \qquad (63)$$

$$t_j(k) := \frac{f_{j-1} \sinh(j+2)\frac{\pi k}{2} - f_{j+1} \sinh j\frac{\pi k}{2}}{f_j} \, g(k). \qquad (64)$$

As usual, $\mathcal{F}_k [t_j(x)] := t_j(k)$. We see that D_j from equation (63) satisfies equation (61) for $j = 1, \ldots, m - 2$. The function $g(k)$ has to be determined from the last equation $j = m - 1$ and is found to be

$$g(k) = \left[2 \cos \frac{\pi}{m+2} \sinh(m+2)\frac{\pi k}{2} \right]^{-1}. \qquad (65)$$

The first equation, $j = 0$, is already contained in equation (63) by $t_0 \equiv 0$ and therefore $D_{j=0} \equiv 0$. Equations (64, 65) imply that the auxiliary functions $\ln Y_j$ behave as

$$\ln Y_j(x \to \infty) \sim \ln Y_j^{(\infty)} + \text{const.} \times e^{-4x/(m+2)}. \qquad (66)$$

The constant is given through $\ln \mathfrak{B}\overline{\mathfrak{B}}$, equations (62, 63), and will be determined numerically below. Combining equations (63, 64) and (65), one finds for the impurity part of the free energy

$$f_i(T, h \ll T_K) = -T \ln \frac{\sin \frac{\pi(2S+1)}{m+2}}{\sin \frac{\pi}{m+2}}$$
$$- \frac{T}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh \frac{\pi k}{2}} D_{2S}(k) \, e^{ik \ln T/T_K} \, dk. \qquad (67)$$

Since $T < T_K$, only negative imaginary values for k are allowed.

The leading T-contribution is given by the singularity closest to the real axis of the integrand in equation (67). For $m \geq 3$, this is a simple pole at $k_m = -4i/(m+2)$, so that

$$f_i(T, h \ll T_K) = -T \ln \frac{\sin \frac{\pi(2S+1)}{m+2}}{\sin \frac{\pi}{m+2}} - T\alpha \left(\frac{T}{T_K} \right)^{\frac{4}{m+2}}$$

$$\alpha = \frac{\left(\sin \frac{4\pi S}{m+2} \sin \frac{2\pi(S+1)}{m+2} - \sin \frac{2\pi S}{m+2} \sin \frac{4\pi(S+1)}{m+2} \right)}{(m+2) \cos \frac{\pi}{m+2} \cos \frac{2\pi}{m+2} \sin \frac{\pi(2S+1)}{m+2}}$$
$$\times \int_{-\infty}^{\infty} e^{\frac{4x}{m+2}} \ln[\mathfrak{B}_m \overline{\mathfrak{B}}_m](x) \, dx. \qquad (68)$$

Table 1. Numerical results for the low temperature Wilson ratios in the over-screened case $m > 2S$. In the isotropic case, we compare our data with Affleck's CFT approach [22], equation (71). For finite anisotropy, the values given in brackets are those from equation (72).

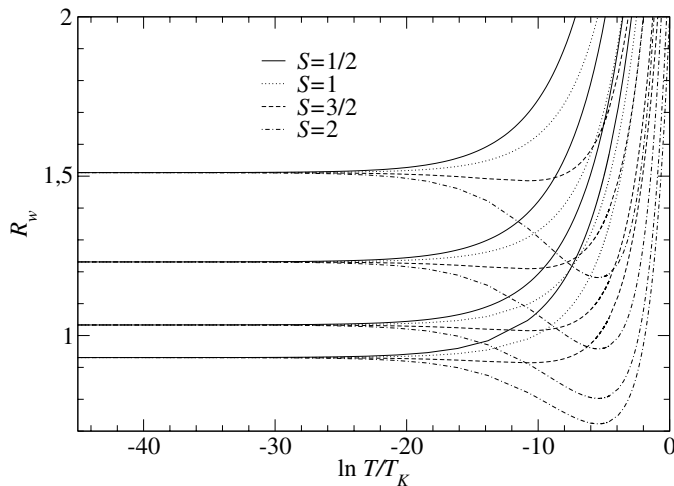| m	$\gamma = 0$	[22]	$\gamma = 0.2$	$\gamma = 0.5$	$\gamma = 0.8$
2	0.202638	0.202642	0.2251582 (0.2251582)	0.2071899 (0.2071898)	0.3377373 (0.3377373)
3	0.370	0.369	0.4102 (0.4104)	0.490 (0.493)	0.607 (0.616)
4	0.60791	0.60793	0.6749 (0.6755)	0.805 (0.811)	0.99 (1.01)
5	0.928	0.931	1.033 (1.034)	1.23 (1.24)	1.51 (1.55)

Fig. 7. The low-temperature Wilon ratios for the overscreened case with $m = 5$ and anisotropy $\gamma = 0$, 0.2, 0.5, $0.8\gamma_{\max}$, where the lowest set of curves is for $\gamma = 0$.

The specific heat and susceptibility are derived:

$$C_i(T, h=0) = \alpha|_{h=0} \frac{4(m+6)}{(m+2)^2} \left(\frac{T}{T_K}\right)^{\frac{4}{m+2}} \qquad (69a)$$

$$\chi_i(T, h=0) = \left(\frac{T}{T_K}\right)^{\frac{4}{m+2}-1} \frac{\partial^2}{\partial(\beta h)^2}\alpha\Big|_{h=0} \qquad (69b)$$

The derivative in equation (69b) only acts upon the integrand in equation (68). Both C_i/T and χ_i show the same T-dependence, yielding the low-temperature Wilson ratio, defined in equation (23):

$$R_w = \frac{\chi_i(T)}{C_i(T)} \frac{C_h(T)}{\chi_h(T)} = \frac{4\pi^2}{3} \frac{(m+2)^2}{4(m+6)} \frac{\frac{\partial^2\alpha}{\partial(\beta h)^2}}{\alpha}\Big|_{h=0} \qquad (70)$$

for $m \geq 3$. The quantity α defined in (68) depends on S, m, γ and βh, which is taken to be zero in (69a, 69b). This means that the α-factors in (69a, 69b) are constant and known, so that from measuring C_i or χ_i, the scale T_K can be determined. So the parameter T_K has the meaning of a low-temperature scale also in the overscreened case.

We did not find an analytical expression of the integral in equation (68), but determined the coefficients in equations (69a, 69b) numerically. Therefore, the NLIE are solved as described in the appendix, and the coefficient of the leading $T^{4/(m+2)}$-decay of $C_i(T \ll T_K)$, $\chi_i(T \ll T_K) \cdot T$ is extracted from the numerical data. In Figure 7 some curves are shown. The Wilson ratio for the isotropic case is

$$R_w|_{\gamma=0} = \frac{(2+m/2)(2+m)^2}{18}; \; m \geq 2. \qquad (71)$$

For finite anisotropy $\gamma \neq 0$, the magnetic field is scaled by the factor $1/(1 - m\gamma/\pi)$ in the NLIE, equations (12, 13). Furthermore, the non-integer powers in (69a, 69b) are determined by the fusion hierarchy, (65), where the anisotropy parameter γ does not enter explicitly. We therefore conjecture that in analogy to the exactly screened case (23) and two-channel overscreened case (60c), the Wilson ratio for finite γ is

$$R_w|_\gamma = \frac{1}{1 - m\gamma/\pi} \frac{(2+m/2)(2+m)^2}{18}; \; m \geq 2. \qquad (72)$$

This means that anisotropy is irrelevant in the renormalization group sense, in accordance with previous treatments of the underscreened case by TBA [8–10]. On the other hand it has been shown in the CFT-approach [24] that a generic anisotropic spin-exchange is relevant. Since the spin-exchange (2b) is a polynomial in the spin-operators, the situation here is different from that in [24]. We leave further investigations on this issue as a future task. In Table 1, our results are compared with equations (71, 72).

4 High temperature evaluation

In this section, the NLIE are analyzed in the limit of high temperatures $T \gg T_K$ by an asymptotic linearization. The fact that contrary to the TBA approach, we deal with a finite set of NLIE, is most useful here. In the isotropic single channel case, our results are farther reaching than those obtained from the TBA equations [13], the treatment of the isotropic multichannel case and of the anisotropic models in this regime is novel.

This section is divided into three parts. In the first part, the isotropic limit $\gamma = 0$ is considered. The second part treats the low-temperature limit of the underscreened case, which is conceptually very similar to the

regime of high temperatures. The anisotropic case is treated in the last part.

4.1 Isotropic case $\gamma = 0$, $h = 0$

In the high temperature regime, the first corrections to the asymptotic value of the free energy are given by the asymptotic corrections of the concerned auxiliary functions for $x \to -\infty$. Corrections are of algebraic and logarithmic-algebraic nature and can be found within an asymptotic linearization scheme. This is also true for $l > m$ in the low-temperature limit, referring to the region $x \to \infty$. In this case, the set of l equations decouples into two sets of m and $l - m$ equations and the high-temperature-linearization is directly transferable.

One makes use of the knowledge of the asymptotic behavior of the auxiliary functions derived in Appendix A to extract the high-temperature behavior of the free energy, the specific heat and magnetic susceptibility in zero field. Employing once again the approximation equation (A.3), this time in the integral of the free energy, we find

$$\lim_{T \gg T_K} f_i(T) = -T \ln(l+1) + \frac{ml(l+2)\pi}{12} \frac{T}{\ln^3 \frac{T}{T_K}} \quad (73a)$$

$$\lim_{T \gg T_K} C_i(T) = \frac{1}{\ln^4 \frac{T}{T_K}} \frac{ml(l+2)\pi^2}{4} \quad (73b)$$

$$\lim_{T \gg T_K} \chi_i(T) = \frac{l(l+2)}{12T} \left(1 - \frac{m}{\ln \frac{T}{T_K}} - \frac{m^2 \ln \ln T/T_K}{2 \ln^2 \frac{T}{T_K}} \right.$$
$$\left. + \frac{4m^3 \phi(m) + m^2/4}{\ln^2 \frac{T}{T_K}} \right). \quad (73c)$$

Values for $\phi(m)$ are given in Table 2 for $m = 1, \ldots, 5$. Let us cite the special case $m = 1$,

$$\phi(m=1) = 0.04707 \pm 2 \times 10^{-7}. \quad (74)$$

Equations (73a–73c) constitute the first calculation of the leading orders of the specific heat and magnetic susceptibility for general m in the framework of an exact solution. The single-channel case $m = 1$ agrees with known TBA-results, [13]. Especially, the coefficient of the $(\ln T/T_K)^{-2}$-decay of the magnetic susceptibility is determined, from which we will calculate Wilson's number, relating the high-temperature to the low-temperature scale.

From equation (73c), we calculate Wilson's number for the underscreened and exactly screened cases, $l \geq m$. Let us first draw our attention to the exactly screened spin-1/2 case, $l = 1 = m$. Wilson [25] obtained for the zero-field spin-1/2 susceptibility by his renormalization group approach

$$\lim_{T \to 0} \chi_i(T) = \frac{1}{2\pi T_K} = \frac{0.1032 \pm 0.0005}{\widetilde{T}_K}$$

$$\frac{\widetilde{T}_K}{T_K} = 2\pi(0.1032 \pm 0.0005) =: 2\pi\xi. \quad (75)$$

This ratio relates the low-temperature scale T_K to the high-temperature scale $\widetilde{T}_K$, defined by absorbing the term $\mathcal{O}(x^{-2})$ in the asymptotic expansion

$$\lim_{T \gg T_K} \chi_i(T) = \frac{1}{2\pi T} \int_{-\infty}^{\infty} \frac{\partial_{\beta h}^2 \left[\ln \mathfrak{B}_1 \overline{\mathfrak{B}}_1 \right](x)}{\cosh\left(x + \ln T/T_K\right)} \mathrm{d}x$$

$$= \frac{1}{2T} \partial_{\beta h}^2 \left[\ln \mathfrak{B}_1 \overline{\mathfrak{B}}_1 \right](x = -\ln T/T_K)$$

$$= b_{1,0}^{(\chi)} + \frac{b_{1,1}^{(\chi)}}{x} + \frac{b_{1,1}^{(\chi)} \ln|x|}{x^2} + \frac{b_{1,2}^{(\chi)}}{x^2} \Bigg|_{x = -\ln T/T_K}$$

$$= b_{1,0}^{(\chi)} + \frac{b_{1,1}^{(\chi)}}{x - \frac{b_{1,2}^{(\chi)}}{b_{1,1}^{(\chi)}}} + \frac{b_{1,l}^{(\chi)} \ln|x|}{x^2} \Bigg|_{x = -\ln T/T_K}. \quad (76)$$

Wilson's number is identified to be

$$2\pi\xi = \exp\left(-\frac{b_{1,2}^{(\chi)}}{b_{1,1}^{(\chi)}}\right). \quad (77)$$

Andrei and Lowenstein [26] carried out a perturbative expansion of the $2S = m = 1$ free energy, both for $h/T \gg 0$ and $T/h \gg 0$. By requiring that in the first case, the result should depend on h/T_h, in the second case on $T/\widetilde{T}_K$, they deduced the ratio $T_h/\widetilde{T}_K$. Moreover, they determined the ratio T_h/T_K from the (conventional) BA. Arguing that the ratios of the energy scales are universal (unlike the scales themselves, which do depend on the cutoff-scheme used), they found by combining their two results (the analytical expression is due to Hewson [27])

$$\xi = \frac{\mathrm{e}^{C+3/4}}{4\pi^{3/2}} = 0.102676\ldots \quad (78)$$

We generalize Wilson's definition (77) to the general spin-S case, in the presence of m channels. A scheme of numerically solving the integral equations which allows for the calculation of the corresponding ratios is given in the appendix. For general l, m, equation (77) reads in the notation of Section 4.1

$$2\pi\xi = \exp\left[-m^2 \left(4\phi(m) + \frac{1}{4}\right)\right],$$

which only depends on m, analogously to the ratio $\widetilde{T}_h/T_K$ for $T = 0$, equations (45, 46). This is in contradiction with [28]. There, the Wilson numbers for S arbitrary, $m = 1$ are calculated. The ratio T_K/T_h for $T = 0$ is found by BA techniques for $\gamma = 0$ and agrees with ours, equation (30) for $\gamma = 0$. $T_h/\widetilde{T}_K$ is found by conventional perturbation theory. The resulting $T_K/\widetilde{T}_K$ depends exponentially on $S(S+1)$. We leave this question to be clarified.

By inserting equation (74), one gets for $m = 1$:

$$\xi = 0.102678 \pm 2 \times 10^{-6}.$$

This result agrees with equations (75, 78). In Table 2, Wilson numbers for the general $l \geq m$ case are given. Note

Table 2. Wilson numbers for $l \geq m$.

m	$\phi(m)$	ξ
1	$0.04707 \pm 2 \times 10^{-5}$	$0.102678 \pm 2 \times 10^{-6}$
2	$0.0502 \pm 1 \times 10^{-4}$	$6.4604 \times 10^{-2} \pm 6 \times 10^{-6}$
3	$0.044 \pm 1.4 \times 10^{-3}$	$4.455 \times 10^{-2} \pm 8 \times 10^{-5}$
4	$0.038 \pm 1.4 \times 10^{-3}$	$3.178 \times 10^{-2} \pm 7 \times 10^{-5}$
5	$0.034 \pm 1.4 \times 10^{-3}$	$2.124 \times 10^{-2} \pm 6 \times 10^{-5}$

that for $2S > m$, the susceptibility at low temperatures can be obtained from that at high temperatures by replacing $l \to (l - m)$. This does not change the value of ξ, which means that in the under-screened cases, only one scale (namely $\widetilde{T}_K$) governs the low- and high-temperature behavior.

The over-screened case $m > l$ is obtained from equations (73a, 73b, 73c) by replacing $m \to l$ and inserting $\ln Y_{j=l}$ and its derivatives, equations (A.13, A.26), into the definitions of the free energy, specific heat and susceptibility. The Wilson numbers are again given in Table 2, where now $m \to l$, $\phi(m) \to \phi(l)$. In analogy to the exactly and under-screened cases, T_K has been identified as a low-temperature scale also for the over-screened case in equations (69a, 69b). From the universality of the low-temperature Wilson ratio, we expect the Wilson number in the over-screened case also to be universal, in analogy to the exactly screened case.

4.2 Low temperature evaluation in the under-screened case

In this section, we make use of the results of Appendix 2, from which one deduces the low-temperature behavior in the under-screened case of the following quantities ($T \ll T_K$):

$$
f_i(T) = -T \ln(l - m + 1)
$$
$$
- \frac{T}{\ln^3 \frac{T_K}{T}} \frac{m\,(l - m)(l - m + 2)\pi}{12}
$$
$$
C_i(T) = \frac{T}{\ln^4 \frac{T}{T_K}} \frac{m(l - m)(l - m + 2)\pi^2}{4}
$$
$$
\chi_i(T) = \frac{(l - m)(l - m + 2)}{12T} \left(1 + \frac{m}{\ln \frac{T_K}{T}} \right.
$$
$$
\left. - \frac{m^2 \ln \ln T_K/T}{2 \ln^2 \frac{T_K}{T}} + \frac{4m^3 \phi(m) + m^2/4}{\ln^2 \frac{T_K}{T}} \right). \quad (79)
$$

The low-temperature behavior for exact screening is given in Section 3.1, where the expected Fermi liquid behavior shows up. Similarly to $T = 0$, a change of sign in the leading corrections to the asymptotic values of the susceptibility is observed, cf. equations (73c, 79). Its physical interpretation has been given in the sequel of equation (48). It applies analogously in this case.

4.3 Anisotropic case, $\gamma \neq 0$

According to equation (8), the anisotropy parameter is restricted to $0 \leq \gamma \leq \frac{\pi}{2n}$, where $n = \max(l, m)$. Since the kernel $\kappa(x)$ decays exponentially in direct space, corrections to $\ln \mathfrak{B}_l^{(\infty)}$, $\ln Y_j^{(\infty)}$ are expected to be exponentially small. Thus it is no longer permitted to replace convolutions with $s(x)$ by algebraic multiplications. Instead, let us write equation (A.1) in direct space, with the same notations as equation (A.2), however including a finite magnetic field from the beginning. First subtract the asymptotes ($\zeta := \beta h/2$),

$$
\lim_{x \ll 0} 2\mathrm{e}^{-\zeta} \frac{\sinh \zeta(l + 1)}{\sinh \zeta l} \delta_l(x) = \lim_{x \ll 0} \left\{ -\mathrm{e}^x \right.
$$
$$
\left. + \left[s * \delta_{l-1} + \kappa * \delta_l - \kappa_- * \overline{\delta}_l \right](x) \right\}. \quad (80)
$$

Contrary to the isotropic case, the driving term $-\mathrm{e}^x$ cannot be neglected since all quantities on the rhs of equation (80) are exponentially small. We did not find a closed solution to equation (80), but determine the exponent of the leading exponential decay. Therefore first note that the equation

$$
\delta_-^{(0)}(x) = -\mathrm{e}^x + \int_{-\infty}^{0} \kappa(x - y)\delta_-^{(0)}(y)\mathrm{d}y
$$

is directly solvable by Wiener-Hopf techniques. This solution relies on the fact that

$$
1 - \mathcal{F}_k[\kappa] = \frac{\sinh \frac{\pi k}{2} \frac{\pi}{\gamma}}{2 \cosh \frac{\pi k}{2} \sinh \frac{\pi k}{2} \left(\frac{\pi}{\gamma} - l \right)}
$$

is factorizable in functions analytical in the upper and lower half planes. The leading decay is $\delta_-^{(0)}(x \ll 0) \sim \mathrm{e}^{\frac{2\gamma}{\pi} x}$. We take the solution $\delta_-^{(0)}$ as an ansatz for δ_-, where ζ is assumed to be small. It is seen that there is no further restriction on the leading decay of δ_-, such that

$$
\delta_-(x \ll 0) \sim \mathrm{e}^{\frac{2\gamma}{\pi} x}.
$$

However, we did not succeed in determining the coefficient. It depends on ζ, especially,

$$
\lim_{x \ll 0} [\delta_- + \overline{\delta}_-](x) \sim (\widetilde{a}_1 + (\beta h)^2 \widetilde{a}_2)\mathrm{e}^{\frac{2\gamma}{\pi} x}
$$
$$
f_i(T \gg T_K) = -T \left[\ln(l + 1) \right.
$$
$$
\left. + (\widetilde{a}_1 + (\beta h)^2 \widetilde{a}_2) \left(\frac{T_K}{T} \right)^{\frac{2\gamma}{\pi}} \right] \quad (81)
$$
$$
C_i(T \gg T_K) \sim \left(\frac{T_K}{T} \right)^{\frac{2\gamma}{\pi}} \quad (82)
$$
$$
\chi_i(T \gg T_K) = \frac{l(l + 2)}{12T} \left(1 - a_2 \left(\frac{T_K}{T} \right)^{\frac{2\gamma}{\pi}} \right). \quad (83)
$$

Note that both C_i and χ_i show similar decays, contrary to the isotropic case.

If $T \ll T_K$, one argues in close analogy to the isotropic case: The decoupling into two independent sets still holds. However, the asymptotic values of the auxiliary functions for $x \to \infty$ are related to their $x \to -\infty$ counterparts by substituting $l \to (l - m)$ *and* scaling $\beta h \to \alpha \beta h$, $\alpha = (1 - m\gamma/\pi)^{-1}$, equation (15). This scaling affects the susceptibility:

$$f_i(T \ll T_K) = -T\left[\ln(l - m + 1) \right.$$
$$\left. + (\tilde{a}_1 + (\beta h \alpha)^2 \tilde{a}_2)\left(\frac{T}{T_K}\right)^{\frac{2\gamma}{\pi}} \right] \qquad (84)$$

$$C_i(T \ll T_K) \sim \left(\frac{T}{T_K}\right)^{\frac{2\gamma}{\pi}} \qquad (85)$$

$$\chi_i(T \ll T_K) = \frac{(l - m)(l - m + 2)\alpha^2}{12T}$$
$$\times \left(1 - a_2 \left(\frac{T_K}{T}\right)^{\frac{2\gamma}{\pi}} \right). \qquad (86)$$

The constants in equations (81–83) differ from those in equations (84–86). For ease of notation, the same symbols have been used. However, if we send $T \to 0$ first with $h \neq 0$ and $l > m$, the rigorous linearization, Section 3.2, is done. Then $\chi \sim h^{-1} \ln^{-2} h/T_K$, equation (48): The T-dependent power-like divergence is replaced by a logarithmic, h-dependent divergence.

5 Conclusion

We presented a novel exact solution to the anisotropic multichannel spin-S Kondo model. The free energy contribution of the impurity is given by a finite set of $(\max[m, 2S]+1)$-many NLIE. By analytical and numerical studies, we confirm and extend known properties of this model.

The low temperature case is characterized by Fermi liquid behavior in the sense that Wilson ratios are defined. However, only in the exactly screened case $l = m$, the limits $T, h \to 0$ commute and $C_i(T)/T$, χ_i approach finite values. These values were calculated by the dilogarithm technique and for $l \geq m = 1$ by the dressed charge formalism, in the framework of a rigorous linearization.

We analysed the ground state for arbitrary anisotropy, spin and channel number and observed non-commutativity of the limits $h, T \to 0$ for models with $l \neq m$. In the underscreened case, if $h = 0$, an asymptotic approach to free $(l - m)/2$ spin asymptotes was recovered for $T \ll T_K$, formally analogous to the $T \gg T_K$ case, Section 4.2. On the other hand, performing first the limit $T \to 0$ while letting $h \neq 0$, a non-integer rest spin occurs for $\gamma \neq 0$ in the underscreened case, equation (43), connected to a quantum critical point [8,9].

We analyzed the low-temperature behaviour of the over-screened models and found non-integer exponents of h if $T = 0$, equation (54) and of T if $h = 0$, equations (69b, 69a). Especially, we determined numerically low-temperature Wilson ratios for arbitrary anisotropy, confirming and extending results by Affleck [22].

At high temperatures, $T \gg T_K$, the impurity spin approaches asymptotically the behavior of a free spin of magnitude $S = l/2$. Corrections to the asymptotic values depend in their amplitude on the channel number m and have been calculated analytically. Especially, Wilson numbers relating low- to high-temperature scales are determined with the help of a numerical solution.

We expect that new insight into the multichannel case is obtained by generalizing the lattice path integral approach proposed in [11] for the isotropic $S = 1/2$, $m = 1$ model. To this aim, R-matrices must be constructed which are invariant under the action of gl(2|1), containing higher dimensional irreps of su(2), [29].

Finally, it should be possible to derive the quantities determined numerically in this work also by analytical methods. This question is left open for future investigations.

Appendix A: Asymptotic linearization

A.1 The region x → −∞

We begin with the asymptotic expansion of $\ln \mathfrak{B}_l$, $\ln \overline{\mathfrak{B}}_l$, $l \geq m$ in the region $x \to -\infty$. The case $m > l$ is obtained therefrom afterwards. Consider the equation for $\mathfrak{B}_l$ in the $h = 0$-case:

$$\ln \mathfrak{b}_l = s * \ln Y_{l-1} + \kappa * \ln \mathfrak{B}_l - \kappa_- * \ln \overline{\mathfrak{B}}_l \qquad (A.1)$$

We shall show that $\ln \mathfrak{B}_l(x)$, $\ln \overline{\mathfrak{B}}_l(x)$ approach their asymptotes $x \to -\infty$ as x^{-3} and calculate the corresponding coefficient $b_3^{(l)}$.

The first $l - 1$ integral equations determine the j-dependence of $\ln Y_j(x)$, $j = 1, \ldots, l-1$. To see this, define

$$\ln Y_j(x) = 2\ln(j + 1) + 2\delta_j(x)$$
$$\ln \mathfrak{B}_l(x) = \ln(l + 1) + \delta_l(x)$$
$$\lim_{x \to -\infty} \delta_j(x) = 0$$
$$\lim_{x \ll 0} \ln y_j(x) = \ln j(j + 2) + 2\frac{(j + 1)^2}{j(j + 2)}\delta_j(x) + \mathcal{O}\left(\delta_j^2\right)$$
$$\lim_{x \ll 0} \ln \mathfrak{b}_l(x) = \ln l + \frac{l + 1}{l}\delta_l(x) + \mathcal{O}\left(\delta_l^2\right). \qquad (A.2)$$

$\ln \overline{\mathfrak{B}}_l$ ($\ln \overline{\mathfrak{b}}_l$) is related to $\ln \mathfrak{B}_l$ ($\ln \mathfrak{b}_l$) by complex conjugation. The crucial approximation is

$$[s * \delta_j](x) \approx \delta_j(x)/2 + \mathcal{O}(\delta_j''(x)). \qquad (A.3)$$

Since s is an exponentially decaying kernel, this approximation is justified for algebraically decaying δ_j. Such an algebraic behavior is indeed expected from the integration

kernel $\kappa(x)$, which itself decays algebraically for $\gamma = 0$. In the anisotropic case $\gamma \neq 0$, this is not true, since $\delta_j(x)$ also decays exponentially. The δ_j satisfy the recurrence relations

$$2\frac{(j+1)^2}{j(j+2)}\,\delta_j(x) = \delta_{j+1}(x) + \delta_{j-1}(x) \qquad \text{(A.4a)}$$

$$\delta_0(x) \equiv 0 \qquad \text{(A.4b)}$$

$$\frac{l+1}{l}\delta_l(x) = \delta_{l-1}(x) + 3(l+1)d(x), \qquad \text{(A.4c)}$$

where in the last line the function $d(x)$ is to be determined and its prefactor has been chosen by convenience. These equations determine δ_j up to a constant factor. Note that from equations (A.4a, A.4b)

$$\delta_j(x) = j(j+2)d(x).$$

Summarizing,

$$\begin{aligned}\ln Y_j(x) &= 2\ln(j+1) + 2j(j+2)d(x) \\ \ln \mathfrak{B}_l(x) &= \ln(l+1) + l(l+1)d(x).\end{aligned} \qquad \text{(A.5)}$$

The functions $\mathfrak{B}_l$, $\overline{\mathfrak{B}}_l$ are related by complex conjugation, the Y_i are real-valued,

$$\ln \mathfrak{B}_l =: B_{1,l} + \mathrm{i}B_{2,l}, \qquad \ln \overline{\mathfrak{B}}_l =: B_{1,l} - \mathrm{i}B_{2,l}. \quad \text{(A.6)}$$

Define the sum and the difference of the integration kernels,

$$\begin{aligned}\kappa^{(s)}(x) &:= \kappa\left(x + \mathrm{i}\frac{\pi}{2}\right) + \kappa\left(x - \mathrm{i}\frac{\pi}{2}\right) = \mathcal{F}_k^{-1}\left[\mathrm{e}^{-\frac{\pi}{2}|k|}\right] \\ \kappa^{(d)}(x) &:= \kappa\left(x + \mathrm{i}\frac{\pi}{2}\right) - \kappa\left(x - \mathrm{i}\frac{\pi}{2}\right) \\ &= \mathcal{F}_k^{-1}\left[\mathrm{e}^{-\frac{\pi}{2}|k|}\frac{\sinh\frac{\pi}{2}k}{\cosh\frac{\pi}{2}k}\right].\end{aligned}$$

Asymptotically,

$$\begin{aligned}\kappa^{(s)}\left(x + \mathrm{i}\frac{\pi}{2}\right) &\sim \frac{1}{2x^2} - \frac{\mathrm{i}\pi}{2x^3}, \ |x| \to \infty, \\ \kappa^{(d)}(x) &\sim -\mathrm{i}\frac{\pi}{2x^3}, \ |x| \to \infty.\end{aligned} \qquad \text{(A.7)}$$

The convolutions with the κ-kernels are written in the following way:

$$\kappa * \ln \mathfrak{B}_l - \kappa_- * \ln \overline{\mathfrak{B}}_l = \kappa_-^{(d)} * B_{1,1} + \mathrm{i}\,\kappa_-^{(s)} * B_{1,2}$$

$$\kappa_{\pm}^{(\nu)}(x) = \kappa^{(\nu)}\left(x \pm \mathrm{i}\frac{\pi}{2}\right), \quad \nu = s,d.$$

The first non-vanishing term in an asymptotic expansion of $B_{l,1}(x)$ around $|x| \to \infty$ is

$$B_{l,1}(x) \sim \ln(l-m+1) + \ln\frac{l+1}{l-m+1}\theta(-x).$$

The θ-function has to be understood asymptotically for large x. This regime is equivalent to small k-values in

Fourier-space, $\mathcal{F}_k[\theta(-x)] = -\mathrm{i}/k + \pi\delta(k)$. In this region around the origin in Fourier space, $\pi k \mathcal{F}_k[\kappa^{(s)}]/2 \sim \mathcal{F}_k[\kappa^{(d)}]$. Thus it follows that

$$\left[\kappa_-^{(d)} * B_{l,1}\right](x) \sim \ln\frac{l+1}{l-m+1}\left(\frac{\pi^2}{4x^3} - \mathrm{i}\frac{\pi}{4x^2}\right). \quad \text{(A.8)}$$

It is useful to define correction terms to the asymptotic behavior for $x \ll 0$:

$$\lim_{x\to-\infty} \ln \mathfrak{B}_l(x) = \ln(l+1) + \frac{b_{1,3}^{(l)}}{x^3} + \mathrm{i}\frac{b_{2,2}^{(l)}}{x^2}. \qquad \text{(A.9)}$$

One then performs the asymptotic expansion

$$\begin{aligned}&\mathrm{i}\left[\kappa_-^{(s)} * B_{l,2}\right](x) \\ &\sim \frac{\mathrm{i}}{x^2}\left(\frac{1}{2}\int_{-\infty}^{\infty} B_{l,2}(x)\,\mathrm{d}x + b_{2,2}^{(l)}\int_{-\infty}^{\infty}\kappa_s(x)\,\mathrm{d}x\right) \\ &\quad - \frac{\pi}{x^3}\left(\frac{1}{2}\int_{-\infty}^{\infty} B_{l,2}(x)\,\mathrm{d}x + b_{2,2}^{(l)}\int_{-\infty}^{\infty}\kappa_s(x)\,\mathrm{d}x\right), \quad \text{(A.10)}\end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty}\kappa^{(s)}(x)\,\mathrm{d}x &= 1 \\ \int_{-\infty}^{\infty} B_{l,2}(x)\,\mathrm{d}x &= \frac{\pi}{2}\left(-(m \pm 10^{-5}) + \ln\frac{l+1}{l-m+1}\right).\end{aligned}$$

The last integral is done numerically with the indicated precision. In the following, we set

$$\int_{-\infty}^{\infty} B_{l,2}(x)\,\mathrm{d}x = \frac{\pi}{2}\left(-m + \ln\frac{l+1}{l-m+1}\right). \qquad \text{(A.11)}$$

Insert equation (A.9) in equation (A.2) and keep only the linear order in δ_l,

$$\ln \mathfrak{b}_l(x) \sim \ln l + \frac{l+1}{l}\left(\frac{b_{1,3}^{(l)}}{x^3} + \mathrm{i}\frac{b_{2,2}^{(l)}}{x^2}\right). \qquad \text{(A.12)}$$

Combining equations (A.8, A.10, A.12) one expands equation (A.1) around $x \to -\infty$:

$$\begin{aligned}\ln l + \frac{l+1}{l}\frac{b_{1,3}^{(l)}}{x^3} &= \ln l + (l-1)(l+1)d(x) \\ &\quad + \frac{\pi^2}{4x^3}\ln\frac{l+1}{l-m+1} - \frac{\pi}{x^3}\left(\frac{1}{2}\int_{-\infty}^{\infty} B_{2,l}(x)\mathrm{d}x + b_{2,2}^{(l)}\right)\end{aligned}$$

$$\begin{aligned}\frac{l+1}{l}\frac{b_{2,2}^{(l)}}{x^2} &= \frac{1}{x^2}\left(-\frac{\pi}{4}\ln\frac{l+1}{l-m+1}\right. \\ &\qquad\left. + \frac{1}{2}\int_{-\infty}^{\infty} B_{2,l}(x)\mathrm{d}x + b_{2,2}^{(l)}\right).\end{aligned}$$

$$B_l^{(m)}(x) = \Delta_+ - \Delta_- \text{sgn}x + \frac{1}{x}(\Delta_+^{(1)} - \Delta_-^{(1)}\text{sgn}x) + \frac{\ln|x|}{x^2}(\Delta_+^{(1)} - \Delta_-^{(1)}\text{sgn}x) + \frac{1}{x^2}(\Delta_+^{(2)} - \Delta_-^{(2)}\text{sgn}x) \qquad \text{(A.19)}$$

$$\left[\kappa * B_l^{(m)}\right](x) = \frac{\Delta_+ - \Delta_- \text{sgn}x}{2} + \frac{1}{2x}\left(\Delta_- + \Delta_+^{(1)} - \Delta_-^{(1)}\text{sgn}x\right) - \left(\Delta_-^{(1)} - \Delta_+^{(l)} + \Delta_-^{(l)}\text{sgn}x\right)\frac{\ln|x|}{2x^2}$$

$$+ \left(\Delta_-^{(1)}\Psi(2) + \Delta_+^{(2)} - \Delta_-^{(2)}\text{sgn}x + \frac{\varepsilon}{2}\right)\frac{1}{2x^2}, \qquad \text{(A.20)}$$

Using equation (A.11), we find:

$$d(x) = \frac{\pi^2}{12}\frac{m}{x^3}, \quad b_{2,2}^{(l)} = -\frac{l\,m\,\pi}{4}$$

$$b_{1,3}^{(l)} = \frac{l(l+2)\,m}{3}\frac{\pi^2}{4},$$

$$\lim_{x\to-\infty}\ln Y_j(x) = 2\ln(j+1) + j(j+2)\frac{m\pi^2}{6x^3},$$

$$\lim_{x\to-\infty}\ln\mathfrak{B}_l(x) = \ln(l+1) - i\frac{m\,l\,\pi}{4\,x^2} + \frac{m\,l(l+2)\pi^2}{12\,x^3}. \qquad \text{(A.13)}$$

Note that the x-dependence of the corrections is determined through the asymptotic behavior of the kernel in the convolutions $\kappa * \ln\mathfrak{B}_l$, $\kappa * \ln\overline{\mathfrak{B}}_l$. The amplitudes follow from the Y-hierarchy.

We proceed with the asymptotic evaluation of $\partial_{\beta h}\ln Y_j(x)$, $\partial_{\beta h}\ln\mathfrak{B}_l(x)$ in the regime $x \ll 0$. The $\ln Y_j$ are symmetric with respect to βh: The system of NLIE remains the same upon replacing $\beta h \to -\beta h$ and substituting $\mathfrak{B}_l$ by $\overline{\mathfrak{B}}_l$. So $\partial_{\beta h}\ln Y_j$ vanishes identically for $h = 0$. Define $B_l^{(m)} := \partial_{\beta h}\ln\mathfrak{B}_l$. Then the only equation which remains is

$$B_l^{(m)}(x) = \left(1 - e^{-\ln\mathfrak{B}_l(x)}\right)\left[\frac{1}{2} + [\kappa * B_l^{(m)}\right.$$

$$\left. -\kappa_- * \overline{B}_l^{(m)}](x)\right]. \quad \text{(A.14)}$$

$B_l^{(m)}$ and $\overline{B}_l^{(m)}$ are related by change of sign and complex conjugation,

$$B_l^{(m)} =: B_{l,1}^{(m)} + iB_{l,2}^{(m)}$$

$$\overline{B}_l^{(m)} = -\left[B_l^{(m)}\right]^* = -B_{l,1}^{(m)} + iB_{l,2}^{(m)}.$$

Below it is shown that the imaginary part of $B_l^{(m)}$ vanishes for $h = 0$, so $B_l^{(m)} \equiv B_{l,1}^{(m)}$ is real valued. We shall determine asymptotic corrections to $B_l^{(m)}$ up to the order $\mathcal{O}\left(x^{-2}\right)$, so that corrections to $\ln\mathfrak{B}_l(\pm\infty)$ (of order $\mathcal{O}\left(x^{-3}\right)$), can safely be neglected. In this approximation,

$$\lim_{x\to-\infty}\left(1 - e^{-\ln\mathfrak{B}_l(x)}\right) = \frac{l}{l+1}\left(1 + i\frac{\pi}{4x^2}\right)$$

$$\lim_{x\to\infty}\left(1 - e^{-\ln\mathfrak{B}_l(x)}\right) = \frac{l-m}{l-m+1}\left(1 + i\frac{\pi}{4x^2}\right), \quad \text{(A.15)}$$

and one finds the asymptotic equation for $B_l^{(m)}$,

$$\lim_{x\ll 0}B_l^{(m)}(x) = \frac{l}{l+1}\left(\frac{1}{2} + \kappa^{(s)} * B_l^{(m)}\right). \qquad \text{(A.16)}$$

The imaginary contributions vanish as expected. An equation analogous to (A.16) does not exist in the TBA-approach, where one deals with the infinitely many Y_j and their derivatives. But $Y_j^{(m)} \equiv 0 \,\forall j$, as mentioned above.

The constant asymptotic behavior of $B_l^{(m)}$ is written in the compact form

$$B_l^{(m)}(x) = \Delta_+ - \Delta_-\text{sgn}x \qquad \text{(A.17)}$$

$$\Delta_+ + \Delta_- = \frac{l}{2} \qquad \Delta_+ - \Delta_- = \frac{l-m}{2}.$$

Equation (A.17) and similar equations in the following have to be understood asymptotically, for $|x| \to \infty$. In a similar manner,

$$\kappa^{(s)} * \text{sgn}x = \text{sgn}x - \frac{1}{x}$$

$$\kappa^{(s)} * \frac{\text{sgn}x}{x} = \frac{\text{sgn}x}{x} + \frac{\ln|x|}{x^2} - \frac{\Psi(2)}{x^2} \qquad \text{(A.18a)}$$

$$\kappa^{(s)} * \frac{1}{x} = \frac{1}{x}, \qquad \kappa^{(s)} * \frac{\ln|x|}{x^2} = \frac{\ln|x|}{x^2}. \qquad \text{(A.18b)}$$

$\Psi(x)$ is the digamma function; $\Psi(2) = -C + 1$, $C = 0.577\ldots$ is Euler's constant. The asymptotic evaluation of the convolutions is done by using distributions. To determine accurately the x^{-2}-coefficient, the precise behavior of the auxiliary functions around the origin must be known. This is done numerically as shown below.

We make the following ansatz for $B_l^{(m)}$, which consists in extrapolating the asymptotic behavior over the whole axis with the aid of distributions:

See equations (A.19, A.20) above,

where ε is determined numerically (cf. Appendix, Eq. (B.3)). The convolutions in the second equation are done with the help of equations (A.18a, A.18b). The terms $\text{sgn}x$ and $\text{sgn}x/x$ give the next higher order contribution when convoluted with the kernel. The asymptotic form of equation (A.16) is found by inserting equations (A.19, A.20). By comparing coefficients, one

finds

$$\Delta_+^{(1)} + \Delta_-^{(1)} = l\,\Delta_- = \frac{m \cdot l}{4} \qquad \Delta_+^{(1)} = \frac{m(2l-m)}{8}$$

$$\Delta_+^{(1)} - \Delta_-^{(1)} = (l-m)\Delta_- = \frac{m(l-m)}{4} \qquad \Delta_-^{(1)} = \frac{m^2}{8}$$

$$\Delta_+^{(l)} + \Delta_-^{(l)} = -l\Delta_-^{(1)} = -\frac{m^2 \cdot l}{8}$$

$$\Delta_+^{(l)} - \Delta_-^{(l)} = -(l-m)\Delta_-^{(1)} = -\frac{m^2(l-m)}{8}$$

$$\Delta_+^{(2)} + \Delta_-^{(2)} = l(\Delta_-^{(1)}\Psi(2) + \varepsilon/2) \tag{A.21}$$

$$\Delta_+^{(2)} - \Delta_-^{(2)} = (l-m)(\Delta_-^{(1)}\Psi(2) + \varepsilon/2). \tag{A.22}$$

In the high-temperature regime, the $x \to -\ln T/T_K \ll 0$ behavior is of importance,

$$\lim_{x \ll 0} B_l^{(m)}(x) = \frac{l}{2} + \frac{m\,l}{4\,x} - \frac{m^2\,l}{8}\frac{\ln|x|}{x^2}$$
$$+ \frac{l}{x^2}\left(\frac{m^2\,\Psi(2)}{8} + \frac{\varepsilon}{2}\right). \tag{A.23}$$

Compare equation (A.23) with the low temperature results equations (47, 48). Both are formally identical up to the order $\ln|x|/x^2$, $x = -\ln T/T_K$ in equation (A.23), $x = -\ln h/\widetilde{T}_h$ in equations (47, 48).

We shall focus on the $x \to +\infty$ asymptotes in the next paragraph.

Our analysis is continued by expanding $\partial_{\beta h}^2 \ln Y_j =: Y_j^{(\chi)}$, $\partial_{\beta h}^2 \ln \mathfrak{B}_l =: B_l^{(\chi)}$, $\partial_{\beta h}^2 \ln \overline{\mathfrak{B}}_l =: \overline{B}_l^{(\chi)}$. These functions are given by the system

$$Y_j^{(\chi)} = \left(1 - e^{-\ln Y_j}\right) s * \left(Y_{j-1}^{(\chi)} + Y_{j+1}^{(\chi)}\right) \tag{A.24a}$$

$$Y_0^{(\chi)} \equiv 0 \tag{A.24b}$$

$$B_l^{(\chi)} = \frac{e^{-\ln \mathfrak{B}_l}}{1 - e^{-\ln \mathfrak{B}_l}}\left[B_l^{(m)}\right]^2 + \left(1 - e^{-\ln \mathfrak{B}_l}\right) s * Y_{l-1}^{(\chi)}$$
$$+ \left(1 - e^{-\ln \mathfrak{B}_l}\right)\left(\kappa * B_l^{(\chi)} - \kappa_- * \overline{B}_l^{(\chi)}\right) \tag{A.24c}$$

$$B_l^{(\chi)} =: B_{l,1}^{(\chi)} + i B_{l,2}^{(\chi)}.$$

The vanishing of $B_2^{(m)}$ helps to expand $B_1^{(\chi)}$:

$$B_{l,1}^{(\chi)} = \frac{1}{l}\left[B_{l,1}^{(m)}\right]^2 + \frac{l}{2(l+1)}Y_{l-1}^{(\chi)}$$
$$+ \frac{l}{l+1}\mathrm{Re}\left(\kappa_-^{(d)} * B_{l,1}^{(\chi)} + i\kappa_-^{(s)} * B_{l,2}^{(\chi)}\right) \tag{A.25a}$$

$$B_{l,2}^{(\chi)} = -\frac{\pi m}{4x^2}\left[B_{l,1}^{(m)}\right]^2 + \frac{lm}{l+1}\frac{\pi}{8x^2}Y_{l-1}^{(\chi)}$$
$$+ \frac{l}{l+1}\mathrm{Im}\left(\kappa_-^{(d)} * B_{l,1}^{(\chi)} + i\kappa_-^{(s)} * B_{l,2}^{(\chi)}\right). \tag{A.25b}$$

From equation (A.25b), $B_{l,2}^{(\chi)} = \mathcal{O}\left(x^{-2}\right)$. Together with equation (A.7), one concludes that the last term in brackets in equation (A.25a) is $\mathcal{O}\left(x^{-3}\right)$. This means that for our purposes, the convolutions in equation (A.25a) can

be entirely neglected. Thus in the asymptotic limit, equations (A.24a–A.24c) are simplified considerably:

$$Y_j^{(\chi)} = \frac{j(j+2)}{2(j+1)^2}\left(Y_{j-1}^{(\chi)} + Y_{j+1}^{(\chi)}\right)$$

$$Y_0^{(\chi)} \equiv 0$$

$$B_l^{(\chi)} = \frac{1}{l}\left[B_l^{(m)}\right]^2 + \frac{l}{2(l+1)}Y_{l-1}^{(\chi)}.$$

This system bears similarity with equations (A.4a–A.4c). From the solution of those equations, we conclude

$$\lim_{x \to -\infty} Y_j^{(\chi)}(x) = \frac{j(j+2)}{6}\left(1 + \frac{m}{x} - \frac{m^2\ln|x|}{2x^2}\right.$$
$$\left. + \left(\frac{m^2}{2}\Psi(2) + \frac{m^2}{4} + 2\varepsilon\right)\frac{1}{x^2}\right) \tag{A.26}$$

$$\lim_{x \to -\infty} B_l^{(\chi)}(x) = b_{l,0}^{(\chi)} + \frac{b_{l,1}^{(\chi)}}{x} + b_{l,2}^{(\chi)}\frac{\ln|x|}{x^2} + \frac{b_{l,2}^{(\chi)}}{x^2}$$
$$= \frac{l(l+2)}{12}\left(1 + \frac{m}{x} - \frac{m^2\ln|x|}{2x^2}\right.$$
$$\left. + \left(\frac{m^2}{2}\Psi(2) + \frac{m^2}{4} + 2\varepsilon\right)\frac{1}{x^2}\right).$$

In Appendix B, a procedure for numerically determining the coefficient $\Delta_+^{(2)} - \Delta_-^{(2)}\mathrm{sgn}x$ of the x^{-2}-decay is described. It deviates slightly from the analytical estimate. We introduce an extra symbol ϕ,

$$\phi(m) := \frac{\left(\Delta_+^{(2)} + \Delta_-^{(2)}\right)}{lm^3} = \frac{1}{m^3}\left(\frac{m^2}{8}\Psi(2) + \frac{\varepsilon}{2}\right). \tag{A.27}$$

Numerically, ϕ is found to be independent of l. However, our data do not suffice to exclude a dependence on m. The results are given in Table 2 in the main body of this work.

If $l < m$, one replaces in the whole preceding analysis of this appendix $l \to m$ and $m \to l$. The thermodynamical quantities are given by Y_l and its derivatives.

A.2 The region x → ∞

We determine the behavior of the auxiliary functions for large values $x \to \infty$ in the under-screened case $l > m$. In this limit, the set of NLIE decouples in two separate sets with exponential accuracy. Consider first the field-free case $h = 0$. The last $l - m$ functions, namely $Y_{m+1}, \ldots, \mathfrak{B}_l, \overline{\mathfrak{B}}_l$, satisfy a set of equations formally identical to equations (A.4a–A.4c), with j, l replaced by $j - m, l - m$. The whole analysis of the preceding paragraph applies to this case; of special interest are now the corrections to the $x \sim \ln\frac{T_K}{T} \gg 0$-behavior of the auxiliary functions. The

results are:

$$\mathfrak{B}_l(x) = \ln(l - m + 1) - \mathrm{i}\frac{m\,(l-m)\,\pi}{4\,x^2}$$
$$+ \frac{m\,(l-m)(l-m+2)\pi^2}{12\,x^3}$$

$$\ln Y_{j>m}(x) = \ln(l+1-j) + \frac{m\,(l-m)(l-m+2)\pi^2}{6\,x^3}$$

$$B_l^{(m)}(x) = \frac{l-m}{2} + \frac{m\,(l-m)}{4\,x} - \frac{m^2\,(l-m)}{8}\frac{\ln|x|}{x^2}$$
$$+ \frac{l-m}{x^2}\left(\frac{m^2\Psi(2)}{8} + \frac{\varepsilon}{2}\right)$$

$$Y_j^{(\chi)}(x) = \frac{(j-m)(j-m+2)}{6}\left(1 + \frac{m}{x} - \frac{m^2\ln|x|}{2x^2}\right.$$
$$\left. + \left(\frac{m^2}{2}\Psi(2) + \frac{m^2}{4} + 2\varepsilon\right)\frac{1}{x^2}\right)$$

$$B_l^{(\chi)}(x) = \frac{(l-m)(l-m+2)}{12}\left(1 + \frac{m}{x} - \frac{m^2\ln|x|}{2x^2}\right.$$
$$\left. + \left(\frac{m^2}{2}\Psi(2) + \frac{m^2}{4} + 2\varepsilon\right)\frac{1}{x^2}\right).$$

Appendix B: Details of the numerical treatment

From the analytical analysis of the NLIE, numerically ill-conditioned terms were found, namely those $\mathcal{O}(1)$, $\mathcal{O}(x^{-1})$ in the limit $|x| \to \infty$ [1]. Since in Fourier space, they would appear as simple poles and discontinuities in the origin, they are subtracted from the auxiliary functions *before* performing the FFT and treated separately. After having applied the inverse FFT, the analytically convoluted terms are re-added:

$$\kappa * \ln Y = \underbrace{\kappa * (\ln Y - f)}_{\text{N}} + \underbrace{\kappa * f}_{\text{A}}, \qquad (B.1)$$

$$\lim_{|x|\to\infty} \ln Y(x) \sim f(x), \qquad (B.2)$$

and κ is a kernel, $\ln Y$ an auxiliary function. The function f is chosen to be asymptotically equal to $\ln Y$, equation (B.2), and contains the numerically ill-conditioned contributions. The difference $\ln Y - f$ is therefore numerically transformable. The term denoted by N is treated numerically, that labeled by A analytically. The Fourier-transforms of f and κ are known. The convolution $k * f$ is solved separately, such that no further manipulations in Fourier space are necessary. We solve numerically for $(\ln Y - f)$: By including terms $\mathcal{O}(1)$, $\mathcal{O}(x^{-1})$ and $\mathcal{O}(\ln|x|/x^2)$ in $f(x)$, the function $(\ln Y - f)$ decays as x^{-2}. Since the kernels also decay as x^{-2}, the coefficient of the x^{-2} decay of $\ln Y$ is obtained from the integral of $(\ln Y - f)$, a quantity which is accessible numerically with high accuracy. Consider the case $\ln Y \equiv B_l^{(m)}$, given by (A.14), and define the regular function $B_{reg} := B_l^{(m)} - f$, such that $B_{reg}(|x| \to \infty) \sim x^{-2}$. Then for $|x| \to \infty$

$$\kappa * B_{reg}(x) \sim \kappa(x)\int_{-\infty}^{\infty} B_{reg}(x)\,\mathrm{d}x$$
$$+ B_{reg}(x)\int_{-\infty}^{\infty} \kappa(x)\,\mathrm{d}x, \quad (B.3)$$

and the quantity ε in equation (A.20) is identified as $\varepsilon := \int_{-\infty}^{\infty} B_{reg}(x)\,\mathrm{d}x$.

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[1] Algebraic corrections to the asymptotes only occur in the isotropic case.